Extra Practice:
Solve the Wave Equation, one free, one fixed

NOTE: It would be better to use d'Alembert's solution (which is covered later). This is only for practice manipulating these series when separating variables.

The PDE we consider is:

$$\begin{array}{rl} \text{PDE} & u_{tt} = c^2 u_{xx} \qquad 0 < x < L \\ \text{BCs} & u(0,t) = 0, \quad u_x(L,t) = 0 \\ \text{ICs} & u(x,0) = f(x) \quad u_t(x,0) = g(x) \end{array}$$

We'll solve using separation of variables:

$$u = XT \quad \Rightarrow \quad XT'' = c^2 X'' T \quad \Rightarrow \quad \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$$

Therefore, splitting the ODEs up, we have:

$T'' + \lambda c^2 T = 0$		$X'' + \lambda X = 0$ $X(0) = 0, X'(L) = 0$
$T(t) = 0$	$\lambda = 0$	$X(x) = 0$
	$\lambda < 0$	$X(x) = C_1 e^{\sqrt{-\lambda}\,x} + C_2 e^{-\sqrt{-\lambda}\,x}$ $X(0) = 0, X'(L) = 0 \quad \Rightarrow \quad X(x) = 0$
	$\lambda > 0$	$X(x) = C_1 \cos(\sqrt{\lambda}\,x) + C_2 \sin(\sqrt{\lambda}\,x)$

Finishing up this last solution in X, T, we see that for X,

$$X(0) = 0 \quad \Rightarrow \quad C_1 = 0$$

For the other end, for $n = 1, 2, 3, \cdots$,

$$X'(L) = 0 \quad \Rightarrow \quad \sqrt{\lambda} C_2 \cos(\sqrt{\lambda}\,L) = 0 \quad \Rightarrow \quad \sqrt{\lambda} L = \frac{(2n-1)\pi}{2}$$

Therefore, the eigenvalues and eigenfunctions are:

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L} \right)^2, \qquad X_n(x) = \sin\left(\frac{(2n-1)\pi}{2L}\, x \right)$$

For time, we have:

$$T'' + \left(\frac{(2n-1)\pi c}{2L} \right)^2 T = 0$$

so that

$$T(t) = A_n \cos\left(\frac{(2n-1)\pi c}{2L}\,t\right) + B_n \sin\left(\frac{(2n-1)\pi c}{2L}\,t\right)$$

The general solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{(2n-1)\pi}{2L}\,x\right)\left(A_n \cos\left(\frac{(2n-1)\pi c}{2L}\,t\right) + B_n \sin\left(\frac{(2n-1)\pi c}{2L}\,t\right)\right)$$

To solve for the coefficients, we go back to the initial conditions:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{(2n-1)\pi}{2L}\,x\right) = f(x)$$

Therefore, these coefficients are the Fourier cosine coefficients for $f(x)$:

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{(2n-1)\pi}{2L}\,x\right)\,dx, \text{ for } n = 1, 2, 3, \cdots$$

Similarly,

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{(2n-1)\pi c}{L} \sin\left(\frac{(2n-1)\pi}{2L}\,x\right) = g(x)$$

Therefore,

$$B_n \frac{(2n-1)\pi c}{2L} = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{(2n-1)\pi}{2L}\,x\right)\,dx, \text{ for } n = 1, 2, 3, \cdots$$