Extra Practice: Exercise 4.4.2

In 4.2, it was shown that the displacement u of a nonuniform string satisfies the PDE:

$$\rho_0 u_{tt} = T_0 u_{xx} + Q$$

where Q represents the vertical component of body force per unit length. If $Q = 0$, the PDE is homogeneous. A slightly different homogeneous equation occurs if $Q = \alpha u$.

1. Show that, if $\alpha < 0$, the body force is restoring (towards $u = 0$), and if $\alpha > 0$, the force tends to push the string farther away from $u = 0$.

   Extra HINT: Look at the how the acceleration and u are related by considering

   $$\rho_0 u_{tt} = \alpha u$$

   SOLUTION: If $\alpha < 0$ and $u > 0$, then u_{tt} is negative- Therefore, u is acting in the opposite direction, and is a restoring force.

   If $\alpha > 0$ and $u > 0$, then u_{tt} is also positive, and the string is pushed away from 0.

2. Assume that ρ_0 and α are functions of x, but that T_0 is constant, and separate variables. Analyze the time-dependent ODE.

   SOLUTION: Let $u = XT$, and

   $$\rho_0(x) X T'' = T_0 X'' T + \alpha(x) X T$$

   Divide both sides by $\rho_0(x) X T$ and we get:

   $$\frac{T''}{T} = \frac{T_0}{\rho_0(x)} \frac{X''}{X} + \frac{\alpha(x)}{\rho_0(x)} = -\lambda$$

   Therefore, the time dependent ODE is $T'' + \lambda T = 0$. The solution to this, as usual, depends on the solutions to the characteristic equation, which depends on λ:

   $$r^2 + \lambda = 0 \quad \Rightarrow \quad r = \pm\sqrt{-\lambda}$$

   The three possible outcomes are:

   - $\lambda = 0$: $T(t) = c_1 + c_2 t$
   - $\lambda < 0$: $T(t) = c_1 \cosh(\sqrt{-\lambda}\, t) + c_2 \sinh(\sqrt{-\lambda}\, t)$
     NOTE: It is OK to use exponential here, but we should try to get used to using hyperbolic sine and cosine- it makes some things a lot easier.
   - $\lambda > 0$: $T(t) = c_1 \cos(\sqrt{\lambda}\, t) + c_2 \sin(\sqrt{\lambda}\, t)$

3. Specialize the previous part and make α, ρ_0, T_0 all constant with the conditions:

   $$\begin{array}{lll} \text{BCs} & u(0,t) = 0 & u(L,t) = 0 \\ \text{ICs} & u(x,0) = 0 & u_t(x,0) = f(x) \end{array}$$

   SOLUTION: Go back and analyze the ODE for X, since we have the full set of boundary conditions for it: $X(0) = 0$, $X(L) = 0$.

   $$\frac{T_0}{\rho_0} X'' + \frac{\alpha}{\rho_0} X = -\lambda X$$

Simplifying a bit, we get the following, and we'll make a substitution to make our work a little clearer:

$$X'' = -\frac{\rho_0}{T_0}\left(\frac{\alpha}{\rho_0} + \lambda\right) X \quad \Rightarrow \quad X'' = -\beta X \quad \Rightarrow \quad X'' + \beta X = 0 \quad X(0) = X(L) = 0$$

Therefore, our only nontrivial solution for X is the periodic solution, with eigenvalues and eigenfunctions:

$$\beta_n = \left(\frac{n\pi}{L}\right)^2 \qquad X_n(x) = \sin\left(\sqrt{\beta_n}\, x\right) = \sin\left(\frac{n\pi}{L}\, x\right)$$

Now going back to T, we have to see if restricting $\beta_n \geq 0$ puts a restriction on λ_n:

$$\beta_n = \frac{\rho_0}{T_0}\left(\frac{\alpha}{\rho_0} + \lambda_n\right) = \left(\frac{n\pi}{L}\right)^2$$

Now, solving for λ_n, we get:

$$\lambda_n = -\frac{\alpha}{\rho_0} + \frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2$$

The problem said to assume $\alpha < 0$, and every other constant is positive, so therefore, $\lambda_n > 0$. Now it's easy to solve the time-dependent ODE:

$$T'' + \lambda_n T = 0 \quad \Rightarrow \quad T(t) = A_n \cos(\sqrt{\lambda_n}\, t) + B_n \sin(\sqrt{\lambda_n}\, t)$$

Taking the superposition:

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos(\sqrt{\lambda_n}\, t) + B_n \sin(\sqrt{\lambda_n}\, t)) \sin\left(\frac{n\pi}{L}\, x\right)$$

To find A_n, B_n, we use the initial position and velocity:

$$u(x,0) = 0 = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}\, x\right) \quad \Rightarrow \quad A_n = \frac{2}{L}\int_0^L 0 \cdot \sin\left(\frac{n\pi}{L}\, x\right)\, dx = 0$$

And differentiating, we get

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n \sqrt{\lambda_n} \cos(\sqrt{\lambda_n}\, t) \sin\left(\frac{n\pi}{L}\, x\right)$$

so that, with $u_t(x,0) = f(x)$, we get the B_n's through the sine series for f:

$$f(x) = \sum_{n=1}^{\infty} B_n \sqrt{\lambda_n} \sin\left(\frac{n\pi}{L}\, x\right) \quad \Rightarrow \quad B_n = \frac{2}{\sqrt{\lambda_n} L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}\, x\right)\, dx$$

4. What are the frequencies of vibration?

   SOLUTION: The n^{th} (circular) frequency is

$$\sqrt{\lambda_n} = \sqrt{-\frac{\alpha}{\rho_0} + \frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2}$$