Extra Practice: Exercise 4.4.3

Consider a slightly damped vibrating string that satisfies:

$$\rho_0 u_{tt} = T_0 u_{xx} - \beta u_t$$

1. Briefly explain why $\beta > 0$.

   Extra HINT: Look at the how the acceleration and u are related by considering

   $$\rho_0 u_{tt} = -\beta u_t$$

   SOLUTION: We see here that with $\beta > 0$, the force of acceleration is acting in the opposite direction as the velocity, so this is *friction*.

2. Assume that ρ_0, T_0 and β are constants, and determine the solution by separation of variables that satisfy the given conditions:

   $$\begin{array}{ll} \text{BCs} & u(0,t) = 0 \quad u(L,t) = 0 \\ \text{ICs} & u(x,0) = f(x) \quad u_t(x,0) = g(x) \end{array}$$

   And assume that β is small, $\beta^2 < 4\pi^2 \rho_0 T_0 / L^2$.

   SOLUTION: Let $u = XT$ as usual, and substitute into the PDE to get

   $$\rho_0 X T'' = T_0 X'' T - \beta X T' \quad \Rightarrow \quad \rho_0 X T'' + \beta X T' = T_0 X'' T$$

   With the boundary conditions, it might be easiest to keep the constants with T. To do that, divide both sides by $T_0 X T$ to get:

   $$\frac{\rho_0 T'' + \beta T'}{T_0 T} = \frac{X''}{X} = -\lambda$$

   Analyzing the spatial ODE first, we get our familiar BVP

   $$X'' + \lambda X = 0, \qquad X(0) = 0 \quad X(L) = 0$$

   so that $\lambda = 0$, $\lambda < 0$ lead us to the trivial solution, and the eigenvalues and eigenfunctions are:

   $$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad X_n(x) = \sin\left(\frac{n\pi}{L}\, x\right)$$

   Now we solve the time-dependent ODE

   $$\rho_0 T'' + \beta T' = -\lambda T_0 T \quad \Rightarrow \quad \rho_0 T'' + \beta T' + \lambda T_0 T = 0$$

   The characteristic equation is, and solve using the quadratic formula:

   $$\rho_0 r^2 + \beta r + \lambda T_0 = 0 \quad \Rightarrow \quad r = \frac{-\beta \pm \sqrt{\beta^2 - 4\lambda T_0 \rho_0}}{2\rho_0}$$

   Using the assumption (and the definition of λ_n), we can show that the roots are complex:

   $$\beta^2 < \frac{4\pi^2 \rho_0 T_0}{L^2} < 4\frac{n^2\pi^2}{L^2}\, \rho_0 T_0 \qquad \text{for } n = 1, 2, 3, \cdots$$

   Therefore, the discriminant is negative, and the roots are complex. We also don't want to get hung up in the notation, so let's make a couple of substitutions:

   $$r = -\frac{\beta}{2\rho_0} \pm \frac{\sqrt{4\lambda_n T_0 \rho_0 - \beta^2}}{2\rho_0}\, i = \gamma \pm \omega_n i$$

(note that ω depends on λ_n, thus the ω_n notation). Using this substitution, the solutions in time are the following, and for future reference, the derivative is given as well:

$$T_n(t) = \mathrm{e}^{\gamma t}\left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right)$$

with

$$T_n'(t) = \gamma \mathrm{e}^{\gamma t}\left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right) + \mathrm{e}^{\gamma t}\left(-A_n\omega_n \sin(\omega_n t) + B_n\omega_n \cos(\omega_n t)\right)$$

Taking the superposition:

$$u(x,t) = \sum_{n=1}^{\infty} \mathrm{e}^{\gamma t}(A_n \cos(\omega_n\, t) + B_n \sin(\omega_n\, t)) \sin\left(\frac{n\pi}{L}\, x\right)$$

To find A_n, B_n, we use the initial position and velocity:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}\, x\right) \quad \Rightarrow \quad A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}\, x\right)\, dx$$

And differentiating, we get the following (the derivative was computed earlier):

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} T_n'(0) \sin\left(\frac{n\pi}{L}\, x\right) = \sum_{n=1}^{\infty} (\gamma A_n + \omega_n B_n) \sin\left(\frac{n\pi}{L}\, x\right)$$

Notice that the A_n have already been computed. With this, we can solve for the B_n:

$$(\gamma A_n + \omega_n B_n) = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\, x\right)\, dx$$

with

$$B_n = -\frac{\gamma}{\omega_n} A_n + \frac{2}{\omega_n L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}\, x\right)\, dx$$

A little commentary: As nasty as this may look, adding friction (β) ended up just multiplying our previous solution(s) by

$$\mathrm{e}^{\gamma t} = \mathrm{e}^{\frac{-\beta}{2\rho_0}\, t}$$

which dampens out the oscillations. Adding the dampening also affected the natural frequencies as well, but this was expected- the same thing happened in Math 244 in our analysis of the mass-spring system:

$$mu'' + \gamma u' + ku = 0$$

where γ was friction, k was the spring constant, m was mass.