An exploration of the Bessel functions as basis functions

We want to examine how it is that J_m(sqrt(\lambda) r) can form a basis with fixed m. In this worksheet, we look at the Bessel function of order 0 (we'll fix the radius to be: 0<r<1 instead of generic "a"):

(i) There are n-1 zeros in the interval from 0 to 1.
(ii) We form a partial expansion of a function to see if it is "close" to the original.

The family of functions is indexed by the zeros of the Bessel function, so we'll get 6 of those first:

```
> k:=5; NumFuncs:=20;
  A:=seq(evalf(BesselJZeros(k,n)),n=1..NumFuncs);
```

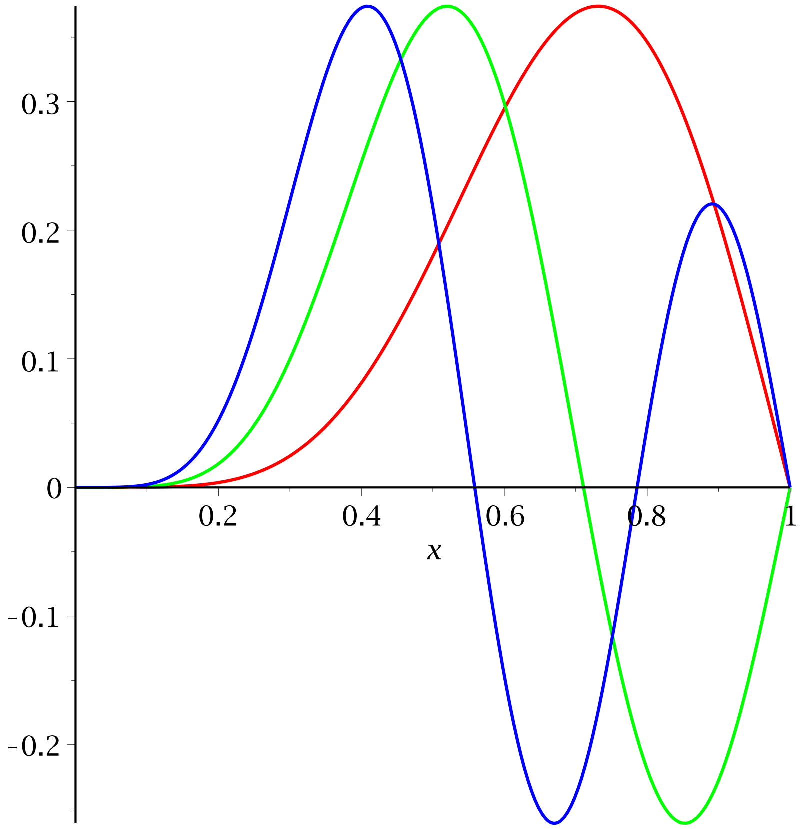
$$k := 5$$

$$NumFuncs := 20$$

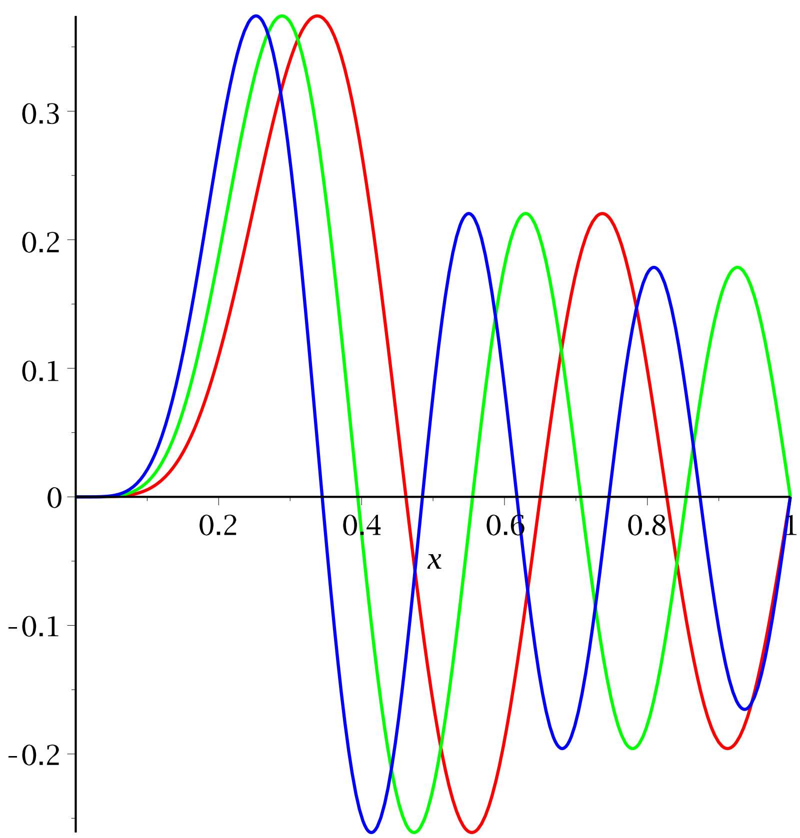
$$A := 8.771483816, 12.33860420, 15.70017408, 18.98013388, 22.21779990, 25.43034115, 28.62661831, 31.81171672, 34.98878129, 38.15986856, 41.32638325, 44.48931912, 47.64939981, 50.80716520, 53.96302656, 57.11730278, 60.27024507, 63.42205405, 66.57289189, 69.72289116 \quad (1)$$

Plot some of the functions, indexed by 1, 2, 3 (then 4, 5, 6)

```
> plot([BesselJ(k,A[1]*x),BesselJ(k,A[2]*x),BesselJ(k,A[3]*x)],x=0..1,color=[red,green,blue]);
```




```
> plot([BesselJ(k,A[4]*x),BesselJ(k,A[5]*x),BesselJ(k,A[6]*x)],x=0.
  .1,color=[red,green,blue]);
```



You'll note from the graph that the nth function has n-1 zeros (do not count the endpoints)

Now we'll compute the first 6 coefficients to a Bessel series expansion. In this example, the function we'll represent is F(r)=sin(2\pi r)

After computing the coefficients, we'll plot the partial sum to see how close it is.

```
> F:=sin(2*Pi*r);
```

$$F := \sin(2\pi r) \quad (2)$$

```
> for m from 1 to NumFuncs do
    Coeffs[m]:=evalf(int(F*BesselJ(k,A[m]*r)*r,r=0..1))/int(
  (BesselJ(k,A[m]*r))^2*r,r=0..1);
  end do;
```

$$Coeffs_1 := -2.311958095$$

$$Coeffs_2 := 1.062749670$$

$$Coeffs_3 := 1.145041091$$

$$Coeffs_4 := 0.9745871540$$
$$Coeffs_5 := 0.7676220278$$
$$Coeffs_6 := 0.6303499151$$
$$Coeffs_7 := 0.5078970456$$
$$Coeffs_8 := 0.4274822289$$
$$Coeffs_9 := 0.3550973874$$
$$Coeffs_{10} := 0.3061939710$$
$$Coeffs_{11} := 0.2607334357$$
$$Coeffs_{12} := 0.2292581105$$
$$Coeffs_{13} := 0.1990528692$$
$$Coeffs_{14} := 0.1777546585$$
$$Coeffs_{15} := 0.1567298020$$
$$Coeffs_{16} := 0.1417080299$$
$$Coeffs_{17} := 0.1265075310$$
$$Coeffs_{18} := 0.1155444566$$
$$Coeffs_{19} := 0.1042068713$$
$$Coeffs_{20} := 0.09597564753 \quad (3)$$

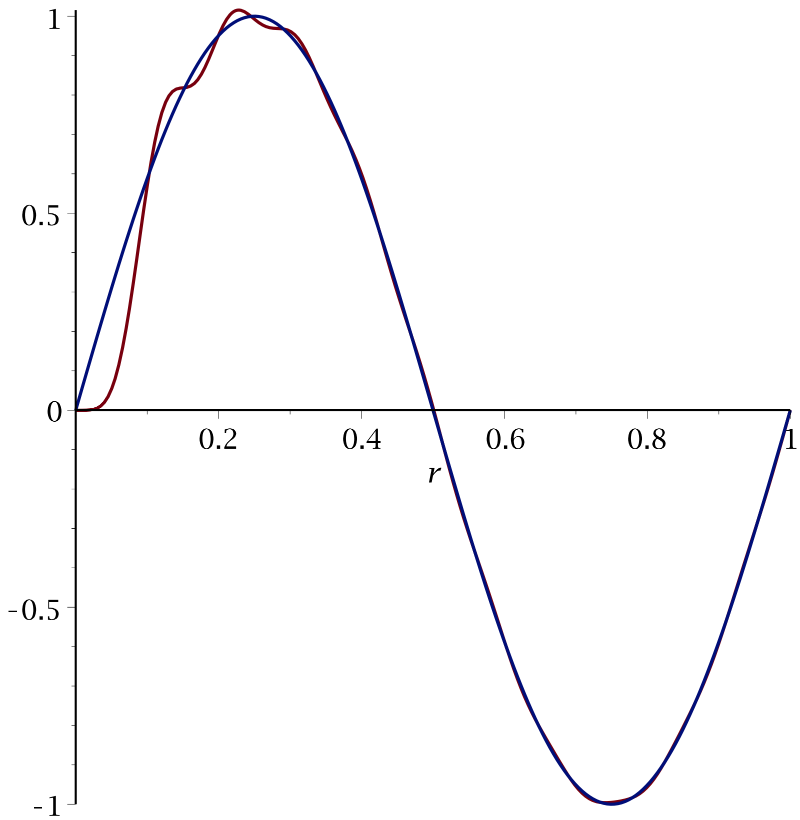
```
> m:='m';
```

$$m := m \quad (4)$$

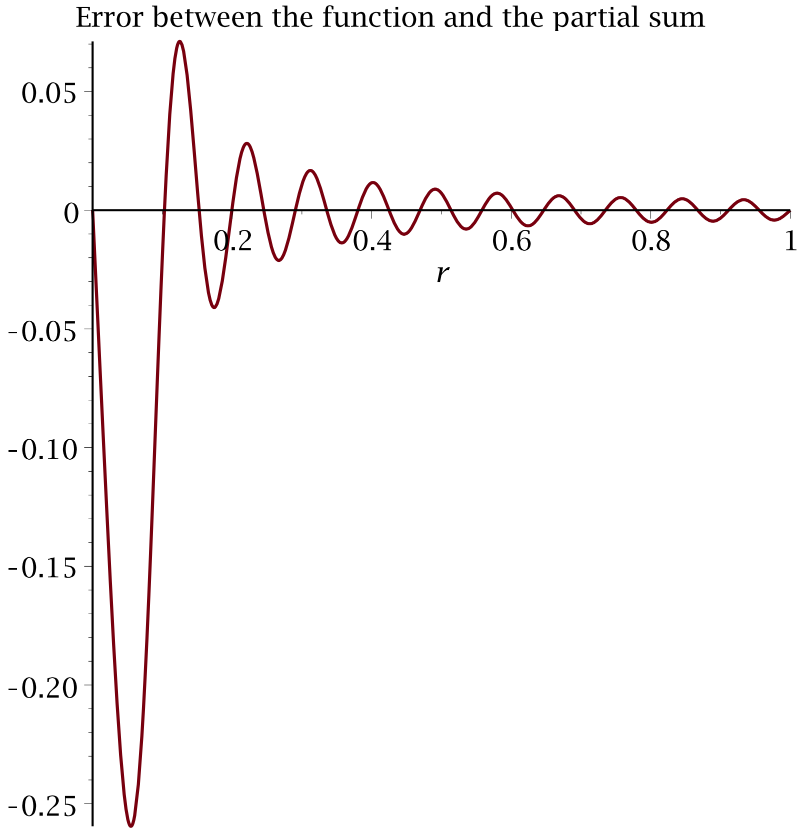
```
> G:=sum(Coeffs[m]*BesselJ(k,A[m]*r),m=1..NumFuncs);
```

$$G := -2.311958095\,\mathrm{BesselJ}(5,\ 8.771483816\ r) + 1.062749670\,\mathrm{BesselJ}(5,\ 12.33860420\ r) + 1.145041091\,\mathrm{BesselJ}(5,\ 15.70017408\ r) + 0.9745871540\,\mathrm{BesselJ}(5,\ 18.98013388\ r) + 0.7676220278\,\mathrm{BesselJ}(5,\ 22.21779990\ r) + 0.6303499151\,\mathrm{BesselJ}(5,\ 25.43034115\ r) + 0.5078970456\,\mathrm{BesselJ}(5,\ 28.62661831\ r) + 0.4274822289\,\mathrm{BesselJ}(5,\ 31.81171672\ r) + 0.3550973874\,\mathrm{BesselJ}(5,\ 34.98878129\ r) + 0.3061939710\,\mathrm{BesselJ}(5,\ 38.15986856\ r) + 0.2607334357\,\mathrm{BesselJ}(5,\ 41.32638325\ r) + 0.2292581105\,\mathrm{BesselJ}(5,\ 44.48931912\ r) + 0.1990528692\,\mathrm{BesselJ}(5,\ 47.64939981\ r) + 0.1777546585\,\mathrm{BesselJ}(5,\ 50.80716520\ r) + 0.1567298020\,\mathrm{BesselJ}(5,\ 53.96302656\ r) + 0.1417080299\,\mathrm{BesselJ}(5,\ 57.11730278\ r) + 0.1265075310\,\mathrm{BesselJ}(5,\ 60.27024507\ r) + 0.1155444566\,\mathrm{BesselJ}(5,\ 63.42205405\ r) + 0.1042068713\,\mathrm{BesselJ}(5,\ 66.57289189\ r) + 0.09597564753\,\mathrm{BesselJ}(5,\ 69.72289116\ r) \quad (5)$$

```
> plot({G,F},r=0..1);
```




```
> plot(G-F,r=0..1,title="Error between the function and the partial
  sum" );
```


Error between the function and the partial sum
0.05
0
-0.05
-0.10
-0.15
-0.20
-0.25
0.2
0.4
0.6
0.8
1
r