A Little Review: Calc 3, Div, Grad and Curl

The Gradient

Given a scalar valued function, $z = f(x, y)$ or $w = f(x, y, z)$, then the **gradient** of f is a vector:

$$z = f(x, y) \quad \Rightarrow \quad \nabla f = \langle f_x, f_y \rangle$$

$$w = f(x, y, z) \quad \Rightarrow \quad \nabla f = \langle f_x, f_y, f_z \rangle$$

Two important facts about the gradient:

- The directional derivative is how we measure the rate of change of f at $\mathbf{x} = \mathbf{a}$ in the direction of the unit vector $\mathbf{w}$: $D_w(f) = \nabla f(\mathbf{a}) \cdot \mathbf{w}$

- The maximum rate of change of f is in the direction of the gradient (and the magnitude of that change is the magnitude of the gradient).

- The gradient vector at a point is orthogonal to the level curve (or level surface) at that point. For example, let $f(x, y) = x^2 + y^2$. The gradient is $\nabla f = \langle f_x, f_y \rangle$. Consider the level curve $f(x, y) = k$. Then the slope of the tangent line to a point on that curve is:

  $$f_x + f_y \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-f_x}{f_y}$$

  Thus, a vector moving in that direction: $\langle 1, -f_x/f_y \rangle$, or $\langle f_y, -f_x \rangle$, and here its easy to see that this is orthogonal to the gradient.

- If $u(x, y)$ is the temperature and $\vec{\phi}$ is the heat flow, then the heat flow is in the opposite direction of the gradient of u. This is Fourier's law extended to multiple dimensions:

  $$\vec{\phi} = -K_0 \nabla u$$

- Often we use operator notation for the gradient: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ where ∇f is understood as $\langle f_x, f_y, f_z \rangle$.

The Laplacian

For shorthand, $\nabla^2 = \nabla \cdot \nabla$. For example, if u is a function of x and y, then:

$$\nabla^2 u = \nabla \cdot (\nabla u) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \cdot \langle u_x, u_y \rangle = u_{xx} + u_{yy}$$

This operation comes up so often, we give it a name: The Laplacian. The notation for it looks like Δ (but it should be clear from the context what we're talking about):

$$\Delta u = \nabla^2 u = u_{xx} + u_{yy}$$

Divergence and Curl

The **divergence** of a vector field $\mathbf{F} = \langle P, Q, R \rangle$ is a scalar quantity:

$$\operatorname{div}(\mathbf{F}) = P_x + Q_y + R_z \quad \text{or} \quad \operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$$

In a fluid flow, the **divergence** is the flux per unit volume, or the rate of change of the density of the fluid.

The **curl** of a vector field is a vector field. The definition is easiest to recall as the cross product:

$$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_x - P_y)\mathbf{k}$$

(Remember the negative sign for the $\mathbf{j}$ term!) The **curl** measures the tendency of the fluid to rotate at that point, the curl points to the axis of rotation, and the magnitude is speed of rotation.

Surface Integrals

We saw that evaluating line integrals $\int_C f(x, y)\, ds$ directly requires us to have a parametric form for the curve, $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ so that $ds = |\mathbf{r}'(t)|\, dt$, and the integral is a standard integral in t.

Similarly, to evaluate a surface integral directly also requires a parametric form for the surface (you might note that a "surface" is a mapping of the plane into three dimensions). The Stewart calculus text uses

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

where $(u, v) \in D$ (domain is D). For example, if $z = f(x, y)$ is the surface of some function, then an obvious way to parametrize it is:

$$\mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle$$

Finally, a couple of facts about surface integrals:

- The area of surface S parametrized by $\mathbf{r}$ over domain D is:

$$\iint_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

  Quick example: Set up a double integral representing the surface area of S, which is the graph of $z = y + x^2$, with domain D: $0 \le x, y \le 1$. Then

$$\mathbf{r} = \langle x, y, y + x^2 \rangle \quad \Rightarrow \quad \mathbf{r}_x = \langle 1, 0, 2x \rangle \quad \mathbf{r}_y = \langle 0, 1, 1 \rangle \quad \mathbf{r}_x \times \mathbf{r}_y = \langle -2x, -1, 1 \rangle$$

Therefore, $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{4x^2+2}$, and the surface area is:

$$\int_0^1 \int_0^1 \sqrt{4x^2+2}\, dy\, dx$$

- The flux of a vector field $\mathbf{F}$ over a surface S with unit normal $\mathbf{n}$:

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

  (Note: $\mathbf{n} = (\mathbf{r}_u \times \mathbf{r}_v)/|(\mathbf{r}_u \times \mathbf{r}_v)|$)

- **Fourier's Law:** If the temperature at (x, y, z) is given by $u(x, y, z)$, then the flux at that point is

$$\boldsymbol{\phi} = -K\nabla u = -K\langle u_x, u_y, u_z\rangle \quad \Rightarrow \quad \nabla\boldsymbol{\phi} = -K\nabla^2 u = -K\langle u_{xx}, u_{yy}, u_{zz}\rangle$$

- **The Divergence Theorem:** Let E be a simple solid region with boundary surface S, with positive orientation. Then:

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint\limits_E \mathrm{div}(\mathbf{F})\, dV$$

- **Example:** Find the flux of the vector field $\mathbf{F} = \langle z, y, x\rangle$ over the unit sphere.

  SOLUTION: The divergence of $\mathbf{F}$:

$$\mathrm{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z = 0 + 1 + 0 = 1$$

  Therefore, the flux of $\mathbf{F}$ across the unit sphere can be computed as a triple integral:

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint\limits_E 1\, dV = \text{ Volume of unit sphere } = \frac{4\pi}{3}$$

Conversion to Polar and Spherical Coordinates

Finally, a quick review of coordinates:

- Polar coordinates: $(x, y) \leftrightarrow (r, \theta)$ where

$$\begin{array}{l|l} x = r\cos(\theta) & r^2 = x^2 + y^2 \\ y = r\sin(\theta) & \tan(\theta) = y/x \end{array}$$

  When integrating, $dxdy$ becomes $r\, dr\, d\theta$

- Cylindrical Coordinates: This coordinate system is polar in the xy plane, and then z is the third coordinate. That is, cylindrical coordinates are:

$$(x, y, z) \leftrightarrow (r, \theta, z)$$

where r, θ are as in the previous section. When integrating, the volume element is $dV = r\,dz\,dr\,d\theta$.

- Spherical Coordinates: A sphere can be defined by angle ϕ (measures from the north pole down), the angle θ is in the $xy-$plane as normal, and its radius ρ. The conversion formulas are:

$$\begin{array}{lll} x = r\cos(\theta) & r = \rho\sin(\phi) & \rho^2 = x^2 + y^2 + z^2 \\ y = r\sin(\theta) & & \cos(\phi) = z/\rho \\ z = \rho\cos(\phi) & & \cos(\theta) = x/r \end{array}$$

When integrating, $dx\,dy\,dz$ is replaced by $\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$

Exercise Set for Section 1.5

1. Show that $u(x, y, t) = 4kt + x^2 + y^2$ solves $u_t = k\nabla^2 u$

2. Show that, if $u(x, t)$ and $v(y, t)$ each solve the one dimensional heat equation (with the same constant k), then $w(x, y, t) = u(x, t)v(y, t)$ will solve the two dimensional heat equation, $u_t = k\nabla^2 u$

3. (*) Verify the divergence theorem directly, by computing the flux integral directly, then comparing it to the volume integral of the divergence, if the vector field is $\mathbf{F} = \langle 0, 0, z\rangle$ and S is the planar surface with corners $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 3)$, and $(1, 1, 3)$. NOTE: The purpose of this question is to review, so open a calc book as necessary.

4. Use the divergence theorem to compute the flux of the vector field:

$$\langle x^2 + y^2, y^2 + z^2, x^2 + z^2\rangle$$

through the cube: $0 \leq x, y, z \leq 1$. NOTE: In this case, use the divergence theorem and don't compute the flux integral directly (you would have to compute it for each side of the cube).

5. (*)(1.5.11 in the text)

6. (1.5.13 in the text- You may assume that we've already shown 1.5.12(c)