Review questions, Chapter 6

1. Find the Laplace transform of the solution the heat equation:

$$\begin{aligned}&u_t = u_{xx}, \quad x > 0, t > 0\\ &u(x,0) = 0\\ &u(0,t) = 1\\ &\lim_{x\to\infty} u(x,t) = 0, \quad t > 0\end{aligned}$$

2. Use the Laplace transform to solve the problem:

$$\begin{aligned}&u_t + 2u_x = 0, \quad x > 0, t > 0\\ &u(x,0) = 3\\ &u(0,t) = 5\end{aligned}$$

3. Use the Laplace transform to solve the wave equation for the transformed solution.

$$\begin{aligned}&u_{tt} = 9u_{xx}, x > 0, t > 0\\ &u(x,0) = 0\\ &u_t(x,0) = 0\\ &u(0,t) = f(t)\\ &lim_{x\to\infty} u(x,t) = 0\end{aligned}$$

4. Use the Laplace transform to find the transform of the solution to:

$$\begin{aligned}&u_t = u_{xx}, \quad x > 0, t > 0\\ &u(x,0) = 0\\ &u_x(0,t) = 1\\ &\lim_{x\to\infty} u(x,t) = 0, \quad t > 0\end{aligned}$$

5. Compute the Fourier sine and cosine transform of e^{-cx}. Hint: You can do them both at once.

6. Find an expression for the Fourier sine transform of $f'(x)$.

7. Find an expression for the Fourier cosine transform of $f''(x)$.

8. Find the transform of the solution (you need to choose sine or cosine) to:

$$\begin{aligned}&y'' - y = \mathrm{e}^{-2x}, \qquad x \geq 0\\ &y(0) = 1\\ &\lim_{x\to\infty} y(x) = 0\end{aligned}$$

9. Find the transform of the solution (you need to choose sine or cosine) to:

$$\begin{aligned}&u_t = u_{xx}, \qquad x > 0, t > 0\\ &u(x,0) = f(x)\\ &u(0,t) = 0\\ &\lim_{x\to\infty} u(x,t) = 0\end{aligned}$$

10. Find the transform of the solution (you need to choose sine or cosine) to:

$$\begin{array}{ll} y'' - y = 3\mathrm{e}^{-4x}, & x \geq 0 \\ y'(0) = 0 & \\ \lim_{x\to\infty} y(x) = 0 & \end{array}$$

11. Find the transform of the solution (you need to choose sine or cosine) to:

$$\begin{array}{ll} u_t = u_{xx}, & x > 0, t > 0 \\ u(x,0) = f(x) & \\ u_x(0,t) = 0 & \\ \lim_{x\to\infty} u(x,t) = 0 & \end{array}$$

12. Find the Fourier transform for the function $f(x) = 1$ if $-1 \leq x \leq 1$, and 0 elsewhere.

13. Find the Fourier transform of $f(x) = \mathrm{e}^{-c|x|}$, $c > 0$.

14. Find the Fourier transform of $f'(x)$ and $f''(x)$ in terms of the Fourier transform of $f(x)$.

15. Show that $\mathcal{F}(f(x-c)) = \mathrm{e}^{-ic\alpha}F(\alpha)$, where $F(\alpha)$ is the Fourier transform of $f(x)$.

16. Find the Fourier transform of the solution for the heat equation below:

$$\begin{array}{ll} u_t = 4u_{xx}, & -\infty < x < \infty, t > 0 \\ u(x,0) = f(x) & \\ \lim_{|x|\to\infty} u(x,t) = 0 & \end{array}$$

17. Find the Fourier transform of the solution for the heat equation below (the sides of the infinite rod are uninsulated):

$$\begin{array}{ll} u_t = u_{xx} - u, & -\infty < x < \infty, t > 0 \\ u(x,0) = f(x) & \\ \lim_{|x|\to\infty} u(x,t) = 0 & \end{array}$$