Review Questions, Exam 2 (Fourier series)

1. True or False (and give a short reason):

   (a) If $f(x)$ is PWS on $[0, L]$, we can find a series representation for f using either sine series or a cosine series.

   SOLUTION: True- either the sine series or the cosine series will converge to $f(x)$ where f is continuous on $[0, L]$.

   (b) If $f(x)$ is PWS on $[-L, L]$, we can find a series representation for f using either sine series or cosine series.

   SOLUTION: False. For the full interval, we'll need both sine and cosines.

   (NOTE: We're assuming that the argument for the functions is the usual $n\pi x/L$, because otherwise the statement could actually be true.)

   (c) If f is PWS on $[-L, L]$, then the sine series for $f(x)$ will converge to the odd extension of f.

   SOLUTION: False- The sine series converges to the odd *part* of f, which was given by

   $$f_o = \frac{1}{2}(f(x) - f(-x))$$

   Of course, if f itself is odd, then it would be true, but then the odd extension is $f(x)$ itself as well.

   (d) The Gibbs phenomenon occurs only when we use a finite number of terms in the Fourier series to represent a function with a jump discontinuity.

   SOLUTION: True. The "ringing" we see only occurs when using a finite sum as an approximation to the infinite sum. In the infinite sum, if f is not continuous at $x = a$, then the Fourier series converges to

   $$\frac{1}{2}(f(a+) + f(a-))$$

   (e) The functions $\sin(nx)$ for $n = 1, 2, 3, \ldots$ are orthogonal to the functions $\cos(mx)$ for $m = 0, 1, 2, \ldots$ on $[0, \pi]$.

   SOLUTION: False. For example,

   $$\int_0^\pi \sin(x)\,dx = -\cos(\pi) + \cos(0) = 2$$

   However, it is true that $\sin(nx)$ and $\sin(mx)$ are orthogonal on $[0, \pi]$ (as are $\cos(nx), \cos(mx)$), or if we extend the interval to $[-\pi, \pi]$, then the statement would be true.

2. What does the Fundamental Theorem of Fourier series say? (be specific and complete!).

SOLUTION: If $f(x)$ is PWS on $[-L, L]$, then the Fourier series

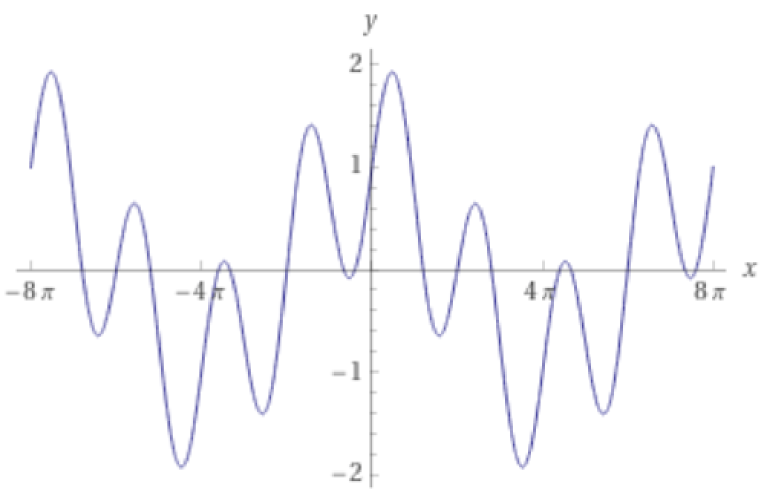
$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

will converge. In addition,

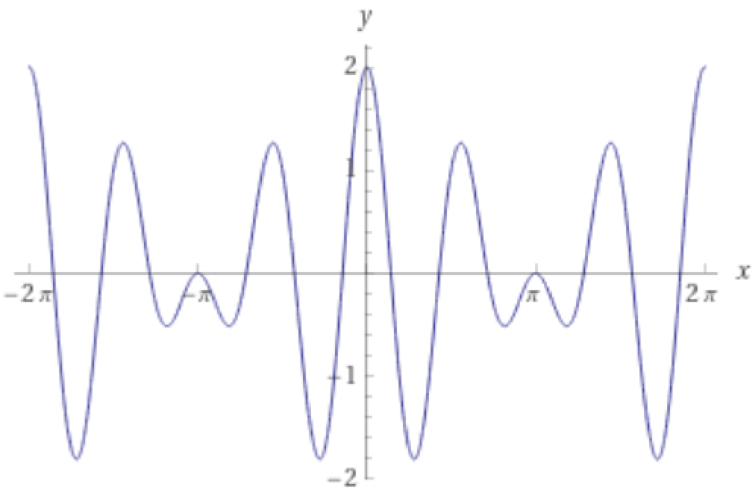
- If $f(x)$ is continuous at x, then the Fourier series converges to $f(x)$.
- If $f(x)$ is discontinuous at x, the the Fourier series converges to $(1/2)(f(x+) + f(x-))$.
- At the endpoints, the Fourier series converges to $(1/2)(f(L-) + f(L+))$

3. Is f periodic (if so, give the period)?

   (a) $f(x) = \cos(x/4) + \sin(x)$
   SOLUTION: The period of $\cos(x/4)$ is 8π and the period of $\sin(x)$ is 2π, so overall, the period is 8π. You can see it here:



   (b) $f(x) = \cos(3x) + \cos(4x)$
   SOLUTION: The periods of the two functions are $2\pi/3$ and $\pi/4$. Thinking about a minimum length, we get to 2π for both.



(c) $f(x) = x\sin(x)$

SOLUTION: This function is not periodic.

4. Is f piecewise continuous (PWC)? Is f piecewise smooth (PWS)?

(a) $f(x) = \begin{cases} x^2 & \text{if } -\pi < x < 0 \\ x^2 + 1 & \text{if } 0 \le x < \pi \end{cases}$

SOLUTION: This function is piecewise continuous (the only point of discontinuity is at zero, and that is a jump discontinuity). The derivative is $2x$ except for a hole at $x = 0$ (the derivative there is not defined, but is just a hole, so the limit exists from the right and left). Therefore, the function is PWC and PWS.

(b) $f(x) = \begin{cases} -\ln(x-1) & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 \le x < 2 \end{cases}$

SOLUTION: The function $\ln(z)$ has a vertical asymptote at $z = 0$, so in this case, there is a vertical asymptote at $x = 1$, so the function is not PWC. Similarly, on $(0, 1)$, the derivative is $-1/(x-1)$, which also has a vertical asymptote at $x = 1$, so the function is not PWS.

(c) $f(x) = \sqrt[3]{x}$ on $[-1, 1]$

SOLUTION: The cube root function is continuous everywhere, so the function is also PWC. The derivative is $\frac{1}{3}x^{-2/3}$, which means we have a vertical asymptote at at $x = 0$, so the function is not PWS.

(d) $f(x) = |x|$ on $[-1, 1]$.

SOLUTION: This function is both continuous (so PWC as well), and PWS.

5. Prove (using the definition), that the product of an odd and even function is odd.

SOLUTION: Let $f(x)$ be odd, and $g(x)$ be even, and let $F(x) = f(x)g(x)$. Then

$$F(-x) = f(-x)g(-x) = -f(x)g(x) = -F(x)$$

6. Show that x^n and x^m are orthogonal on $[-L, L]$ (using the usual inner product, and assuming n, m are positive integers) if n, m are not both even or both odd.

   SOLUTION: Two functions are orthogonal if the inner product is zero. In the "usual" case:

$$\int_{-L}^{L} x^n x^m \, dx$$

   The integral will be zero if $x^n x^m$ is odd, which only happens if n is odd and m is even, or if n is even and m is odd.

7. True or False? (If false, give an example) Assume f is PWS on $[-L, L]$.

   (a) If f is continuous, so is the Fourier series of f.

   SOLUTION: False. If the periodic extension of f was continuous, then the Fourier series would be continuous.

   (b) If f is discontinuous, so is the Fourier series of f.

   SOLUTION: False. It depends on what kind of discontinuity- If the function is not continuous at $x = a$, for example, then the Fourier series converges to $(f(a+) - f(a-))/2$, so if the limits are the same, the Fourier series could be filling in the "hole" at $x = a$. If the discontinuity is a jump discontinuity, then the Fourier series would also be discontinuous there.

8. Draw the Fourier sine series for the function (showing at least three periods):

$$f(x) = \begin{cases} 3 & \text{if } x = 0 \text{ or } x = 1 \\ x + 1 & \text{otherwise, } 0 < x \leq 2 \end{cases} \Bigg]$$

9. Draw the Fourier cosine series of the function in the previous problem (showing at least three periods).

10. Let $f(x) = 3x + 5$. Compute the even and odd parts of f.

    SOLUTION: The odd part is $f_{\text{odd}} = \frac{1}{2}(f(x) - f(-x)) = 3x$

    The even part is $f_{\text{even}} = \frac{1}{2}(f(x) + f(-x)) = 5$

    Side note: If we had the full Fourier series for $3x + 5$ on the interval $[-L, L]$, then the sine series would converge to $3x$ and the cosine series to 5 (in fact, the cosine series *is* just the number 5).

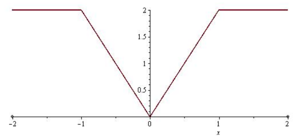
11. Let

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 2 & \text{for } 1 < x < 2 \end{cases}$$

    (a) Write the even extension of f as a piecewise defined function.

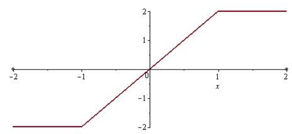
    The even extension of f on the interval $[-2, 2]$ would be defined as:

$$f(x) = \begin{cases} 2 & \text{for } -2 < x < -1 \\ -2x & \text{for } -1 < x < 0 \\ 2x & \text{for } 0 < x < 1 \\ 2 & \text{for } 1 < x < 2 \end{cases}$$

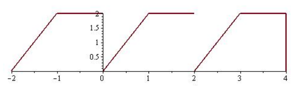


(b) Write the odd extension of f as a piecewise defined function.
Similarly, the odd extension on $[-2, 2]$ is defined as:

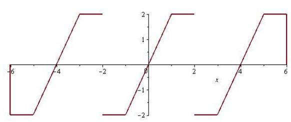
$$f(x) = \begin{cases} -2 & \text{for } -2 < x < -1 \\ 2x & \text{for } -1 < x < 0 \\ 2x & \text{for } 0 < x < 1 \\ 2 & \text{for } 1 < x < 2 \end{cases}$$



(c) Draw a sketch of the periodic extension of f.
SOLUTION:



(d) Find the Fourier sine series (FSS) for f, and draw the FSS on the interval $[-4, 4]$.



NOTE: The vertical lines don't belong in the graph, and in the places where there is a jump discontinuity (at -6, -2, 2, 6), we ought to draw a point to indicate that the series converges to zero there.
The algebraic form of the series is:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \quad \Rightarrow \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

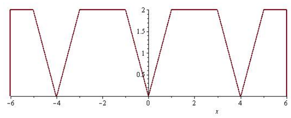
Therefore, with $L = 2$:

$$b_n = \int_0^1 2x\sin\left(\frac{n\pi x}{2}\right)\,dx + \int_1^2 2\sin\left(\frac{n\pi x}{2}\right)\,dx =$$

$$-\frac{4}{n^2\pi^2}(-2\sin\left(\frac{n\pi}{2}\right) + n\pi\cos\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi}(-1 + (-1)^n)$$

It is possible to simplify that a bit, but that is unnecessary for the exam.

(e) Find the Fourier cosine series (FCS) for f, and draw the FCS on the interval $[-4, 4]$.
SOLUTION:



NOTE: The vertical lines don't belong in the graph, the series would continue out in a continuous fashion.

The algebraic form of the series is:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) \quad \Rightarrow \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

The computation for a_0 is slightly different, so do that one first:

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \int_0^1 2x\,dx + \int_1^2 2\,dx = 3$$

And, for $n = 1, 2, 3, \ldots$:

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)\,dx = \int_0^1 2x\cos\left(\frac{n\pi x}{2}\right)\,dx + \int_1^2 2\cos\left(\frac{n\pi x}{2}\right)\,dx$$

For the first integral, we integrate by parts:

$$\begin{array}{ccc} + & 2x & \cos(n\pi x/2) \\ - & 2 & (2/n\pi)\sin(n\pi x/2) \\ + & 0 & -(4/n^2\pi^2)\cos(n\pi x/2) \end{array} \quad \Rightarrow \quad \left.\left(\frac{4x}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \frac{8}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right)\right|_0^1$$

For the first integral, we get

$$\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{8}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{8}{n^2\pi^2}$$

For the second integral, we get

$$\left.\left(\frac{4}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right)\right|_1^2 = 0 - \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

It is possible to simplify that a bit, but that is unnecessary for the exam.

12. Let $f(x)$ be given as below.

$$f(x) = \begin{cases} x \text{ if } -1 < x < 0 \\ 1 + x \text{ if } 0 < x < 1 \end{cases}$$

(a) Find the Fourier series for f (on $[-1, 1]$), and draw a sketch of it on $[-3, 3]$.

SOLUTION: I'll leave the sketch to you. The main purpose here is to have you recall the formulas for the series coefficients. In this case,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

with the formulas:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \int_{-1}^{0} x\,dx + \int_0^1 (1+x)\,dx = 1$$

and, we should find that the a_n's are zero:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)\,dx = \int_{-1}^{0} x\cos(n\pi x)\,dx + \int_0^1 (1+x)\cos(n\pi x)\,dx = 0$$

As we did for the a_n, we'll need to use integration by parts:

$$b_n = \frac{1}{L}\int_{-1}^{1} f(x)\sin(n\pi x/L)\,dx = \int_{-1}^{0} x\sin(n\pi x)\,dx + \int_0^1 (1+x)\sin(n\pi x)\,dx$$

$$\begin{array}{ccc} + & x & \sin(n\pi x) \\ - & 1 & -(1/n\pi)\cos(n\pi x) \\ + & 0 & -(1/n^2\pi^2)\sin(n\pi x) \end{array} \quad \Rightarrow \quad \left(-\frac{x}{n\pi}\cos(n\pi x) + \frac{1}{n^2\pi^2}\sin(n\pi x)\right|_{-1}^{0} =$$

$$(0+0) - (\frac{1}{n\pi}\cos(n\pi) - 0) = -\frac{1}{n\pi}(-1)^n$$

and for the other integral,

$$\left(-\frac{1+x}{n\pi}\cos(n\pi x) + \frac{1}{n^2\pi^2}\sin(n\pi x)\right|_0^1 = \left(-\frac{2}{n\pi}(-1)^n + 0\right) - \left(-\frac{1}{n\pi} + 0\right)$$

Putting them all together:

$$\frac{1}{n\pi}(1 - (-1)^n - 2(-1)^n)$$

(NOTE: If you subtracted 1/2 from your function $f(x)$, it becomes an odd function- That's why the cosine terms ended up being zero).

(b) Find the Fourier sine series for f on $[0,1]$ and draw a sketch of it on $[-3,3]$.

SOLUTION: Again, the main point here is to have you recall the formulas and set up the integrals:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

with

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx = 2\int_0^1 (1+x)\sin(n\pi x)\,dx = 2\frac{1-2(-1)^n}{n\pi}$$

(c) Find the Fourier cosine series for f on $[0, 1]$ and draw a sketch of it on $[-3, 3]$.
SOLUTION: The formulas:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

with

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = 2\int_0^1 (1+x)\,dx = 3$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(n\pi x/L)\,dx = 2\int_0^1 (1+x)\cos(n\pi x)\,dx = 2\frac{(-1)+(-1)^n}{n^2\pi^2}$$

13. For the coefficients, the term $2/L$ or $1/L$ comes from the denominator:

$$\frac{\langle f(x), y_n\rangle}{\langle y_n, y_n\rangle}$$

So for example, in the full Fourier series, we have the term $1/L$ in front, meaning that the denominator evaluates to L. We show that for the cosines:

$$\int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right)\,dx = \frac{1}{2}\int_{-L}^{L} 1 + \cos\left(\frac{2n\pi x}{L}\right)\,dx = \frac{1}{2}\left(x + \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right)\Bigg|_{-L}^{L} = L$$

(You only need to show one of these).