Topics for Exam 3

Exam 3 will cover material from 4.1-4.4, 5.1-5.3. You won't be able to use notes on this exam, however, the summary of eigenfunctions will be provided.

Overview:

In 4.1-4.3, we solve the "big three" PDEs- The heat equation, the wave equation and Laplace's equation- all using our Fourier series. In 4.4, we consider two types on nonhomogeneity- in the BCs or in the PDE itself.

Most of chapter 4 is computation- there's not much theory here.

- Some review items:
  - For convenience, we can use the hyperbolic sine and cosine:

    $$y'' + \omega^2 y = 0 \qquad\qquad y'' - \omega^2 y = 0$$
    $$y(t) = C_1 \cos(\omega t) + D_1 \sin(\omega t) \qquad\qquad y(t) = C_1 \cosh(\omega t) + D_1 \sinh(\omega t)$$

  - The chain rule: If U is a function of ξ, η and each of ξ, η are functions of x and y, then:

    $$u_x = u_\xi \xi_x + u_\eta \eta_x$$
    $$u_y = u_\xi \xi_y + u_\eta \eta_y$$

  - What is the solution to $y' = ay$, $y(0) = y_0$?

    $$y(t) = y_0 \mathrm{e}^{at}$$

  - What is the solution to $y' = ay + b$, $y(0) = y_0$ (where a, b are constant)?

    $$y(t) = (y_0 + b/a)\mathrm{e}^{at} - \frac{b}{a} \quad \text{or} \quad \frac{b}{a}(\mathrm{e}^{at} - 1) + y_0 \mathrm{e}^{at}$$

Section discussions

In section 4.1, we solved the homogeneous heat equation using our eigenfunction technique.

In section 4.2, we solved the homogeneous wave equation. We also discussed **modes of vibration**, **nodes** and the **fundamental mode**. We saw that the graph of a typical solution looked like a standing wave (with the ends tied down).

In section 4.3, we continued solving homogeneous PDEs by looking at solutions to Laplace's equation on a rectangle. We also discussed how we could piece together solutions to get a solution to the PDE with nonhomogeneous boundary conditions (Exercise 12(a)).

In section 4.4, we discussed two types of nonhomogeneity: BCs and the PDE. If we have nonhomogeneous BCs, use a helper function $v(x, t)$ (or just $v(x)$) to satisfy those, then take the overall solution to be $u(x,t) = w(x,t) + v(x,t)$. Be able to show that w now has homogeneous BCs (although using the helper function may make the PDE nonhomogeneous).

The last section of 4.4 discussed PDEs with nonhomogeneous terms- for example, $u_t = u_{xx} + F(x,t)$. In this case, we use an eigenfunction series (typically in X). We then construct eigenfunction series for $F(x,t)$ and $u(x,t)$. Substitute all these series into the PDE, and by equating coefficients, we obtained an (infinite) system of ODEs. We finish by constructing solutions to the ODEs.

In section 5.1, we solve first order PDEs with constant coefficients:

$$au_x + bu_y = 0$$

We did this two ways- One method used the idea that $au_x + bu_y$ can be seen as the dot product of (a,b) with ∇u. This means that $u(x,y)$ is constant in the direction of the lines going in the direction (a,b). Finding the equations of the lines gave us the **characteristic curves** (curves where u is constant). In this case, we could write the lines as:

$$-bx + ay = C_1 \quad \text{or } bx - ay = C_2$$

Then we showed that our solution was in the form $u(x,y) = g(-bx + ay)$ (g is an abitrary function).

The other method was what our textbook used: A change of variables, which will be useful in other sections. The idea is that we'll construct a change of variables, ξ, η each as functions of x, y, so that the PDE in ξ, eta gives a single derivative (rather than two). If we assume $u(\xi, eta)$, then by the chain rule we have:

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x \\ u_y &= u_\xi \xi_y + u_\eta \eta_y \end{aligned}$$

so making the substitution into our PDE, we get:

$$(a\xi_x + b\xi_y)u_\xi + (a\eta_x + b\eta_y)u_\eta = 0$$

We can construct ξ and η so that one of these parenthetical expressions is zero, and the other is not. A simple choice is:

$$\begin{aligned} \xi &= x \\ \eta &= -bx + ay \end{aligned} \qquad \Rightarrow \qquad au_\xi = 0 \text{ or just } u_\xi = 0$$

In the slightly more complicated case, $au_x + bu_y + cu = d$, we can use the same change of coordinates to get:

$$au_\xi + cu = d$$

Think of this as $ay' + cy = d$, which we know how to solve. Notice that there is an existence and uniqueness theorem at the end of 5.1! I won't ask you about it, but it's nice to know that it is there.

For section 5.2, we get slightly more complicated:

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$$

In this case, the change of coordinates takes a bit of a different form. In this case, think of the DE on the left, that hopefully we can solve to get the solution on the right.

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} \quad \Rightarrow \quad h(x,y) = C$$

This leads us to the change of coordinates:

$$\begin{aligned} \xi &= x \\ \eta &= h(x,y) \end{aligned}$$

Section 5.3 is quite different than 5.1, 5.2, so we might separate it out if we run out of time before Exam 3 (to be announced). In this section, we exam D'Alembert's solution to the wave equation- the key difference is that the domain for x is no longer an interval, but the entire real line $-\infty < x < \infty$ (with $t \geq 0$). When we have a spatial domain like this, we can no longer use the eigenfunctions that worked so well in Chapter 4. However, we will see characteristics pop up. In addition, we'll discuss the concepts of "domain of dependence" and "region of influence".