HW: Section 2.6, #10

We'll look at part (b), where we want to solve Laplace's equation:

$$\begin{aligned} &u_{xx} + u_{yy} = 0 \\ &u(x,0) = \sin(3x) \\ &u(x,1) = \sin(x) \\ &u(0,y) = u(\pi,y) = 0 \end{aligned}$$

Using the product solution $u(x,y) = X(x)Y(y)$,

$$X''Y + XY'' = 0 \quad \frac{X''}{X} + \frac{Y''}{Y} = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

With an eye on our boundary conditions,

$$u(0,y) = 0 \quad \Rightarrow \quad X(0)Y(y) = 0 \quad \Rightarrow X(0) = 0$$

and similarly, we'll take $X(\pi) = 0$. Now we have the ODEs:

$$Y'' - \lambda Y = 0 \qquad \begin{aligned} &X'' + \lambda X = 0 \\ &X(0) = 0, X(\pi) = 0 \end{aligned}$$

We'll use the fact that we know the eigenvalues and eigenvectors to the ODE in X:

$$\lambda_n = \frac{n^2\pi^2}{\pi^2} = n^2 \qquad X_n(x) = \sin(nx)$$

Now we solve the ODE in Y using the λ_n:

$$Y'' - n^2 Y = 0 \quad \Rightarrow \quad r^2 = n^2 \qquad r = \pm n$$

At this point, you could use either the hyperbolic sine and cosine (**that's the solution to the last problem of the exam review**), or you could use exponentials. Let's try using the exponentials:

$$Y_{n_1} = \mathrm{e}^{ny} \qquad Y_{n_2} = \mathrm{e}^{-ny}$$

Therefore, the full solution so far is:

$$u(x,y) = \sum_{n=1}^{\infty} C_n \mathrm{e}^{ny} \sin(nx) + D_n \mathrm{e}^{-ny} \sin(nx)$$

Now, if $u(x,0) = \sin(3x)$ and $u(x,1) = \sin(x)$, then we have to determine C_1, D_1, C_3, D_3 and set everything else to zero. Apply our BCs:

$$u(x,y) = C_1 \mathrm{e}^{y} \sin(x) + C_3 \mathrm{e}^{3y} \sin(3x) + D_1 \mathrm{e}^{-y} \sin(x) + D_3 \mathrm{e}^{-3y} \sin(3x)$$

$$u(x,0) = \sin(3x) \quad \Rightarrow \quad C_1 \sin(x) + C_3 \sin(3x) + D_1 \sin(x) + D_3 \sin(3x) = \sin(3x)$$

Therefore, $C_1 + D_1 = 0$ and $C_3 + D_3 = 1$. Going on to the next BC,

$$u(x,1) = \sin(x) \quad \Rightarrow \quad C_1 e \sin(x) + C_3 e^3 \sin(3x) + D_1 e^{-1} \sin(x) + D_3 e^{-3} \sin(3x) = \sin(x)$$

from which we get: $C_1 e + C_1 e^{-1} = 1$ and $C_3 \mathrm{e}^3 + D_3 \mathrm{e}^{-3} = 0$.

Put these together to solve for C_1, D_1 and C_3, D_3 (use Cramer's Rule):

$$\begin{aligned} C_1 + D_1 &= 0 \\ C_1 \mathrm{e} + D_1 \mathrm{e}^{-1} &= 1 \end{aligned} \quad \Rightarrow \quad C_1 = \frac{\begin{vmatrix} 0 & 1 \\ 1 & \mathrm{e}^{-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ \mathrm{e} & \mathrm{e}^{-1} \end{vmatrix}} = \frac{-1}{\mathrm{e}^{-1} - \mathrm{e}}, \quad D_1 = \frac{\begin{vmatrix} 1 & 0 \\ \mathrm{e} & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ \mathrm{e} & \mathrm{e}^{-1} \end{vmatrix}} = \frac{1}{\mathrm{e}^{-1} - \mathrm{e}}$$

Similarly, for C_3, D_3, we have:

$$\begin{aligned} C_3 + D_3 &= 1 \\ C_3 \mathrm{e}^3 + D_3 \mathrm{e}^{-3} &= 0 \end{aligned} \quad \Rightarrow \quad C_3 = \frac{\begin{vmatrix} 1 & 1 \\ 0 & \mathrm{e}^{-3} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ \mathrm{e}^3 & \mathrm{e}^{-3} \end{vmatrix}} = \frac{\mathrm{e}^{-3}}{\mathrm{e}^{-3} - \mathrm{e}^3}, \quad D_3 = \frac{\begin{vmatrix} 1 & 1 \\ \mathrm{e}^3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ \mathrm{e}^3 & \mathrm{e}^{-3} \end{vmatrix}} = \frac{-\mathrm{e}^3}{\mathrm{e}^{-3} - \mathrm{e}^3}$$