Introduction to Chapter 4

The big question we want to ask: If $f(x)$ is represented by its Fourier series as $F(x)$, do we get the Fourier series of $f'(x)$ by differentiating $F(x)$? Let's be more specific with an example:

First we compute the Fourier sine series for $f(x) = x$ on $[0, 1]$, then we differentiate both sides with respect to x:

$$x \sim \sum_{n=1}^{\infty} B_n \sin(n\pi x) \quad \Rightarrow \quad 1 \sim \sum_{n=1}^{\infty} B_n n\pi \cos(n\pi x)$$

We'll show below that $B_n = \frac{2(-1)^{n+1}}{n\pi}$, but substituting it in now, we should get the Fourier series for the derivative.

$$1 \sim \sum_{n=1}^{\infty} 2(-1)^{n+1} \cos(n\pi x)$$

There are several problems with this- The first is that the Fourier cosine series of the derivative $f'(x) = 1$ should be $F'(x) = 1$. The second thing is that the terms of the sum on the right do not go to zero as $n \to \infty$, so the sum does not even converge (by the way, at $x = 1$, the sum equals 0). What happened? We'll discuss that below, but first let's include the details that we left off previously.

Computing B_n

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx = 2 \int_0^1 x \sin(n\pi x)\, dx$$

Integrate by parts:

$$\begin{array}{ccc} + & x & \sin(n\pi x) \\ - & 1 & -\cos(n\pi x)/(n\pi) \\ + & 0 & -\sin(n\pi x)/(n^2\pi^2) \end{array}$$

$$\left(-\frac{x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2} \right|_0^1 = \left(\frac{-\cos(n\pi)}{n\pi} + 0 \right) + (0 - 0) = \frac{-\cos(n\pi)}{n\pi}$$

Using $\cos(n\pi) = (-1)^n$, we then get

$$B_n = \frac{2(-1)^{n+1}}{n\pi}$$

so that the Fourier series for $f(x) = x$ is given by:

$$x \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

Back to the Derivative

If $f(x)$ has a sine series, what should the series of its derivative be? Let's examine that closer:

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad \text{where} \qquad B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

We can compare the two forms for the derivative- One by differentiating term by term, the other by looking at the cosine series for $f'(x)$. (Remember that our goal is to understand the relationship between B_n and A_n in the series below).

- If we differentiate the sine series term-by-term, we get:

$$f'(x) \sim \sum_{n=1}^{\infty} \frac{n\pi}{L} B_n \cos\left(\frac{n\pi x}{L}\right)$$

- If we look at the cosine series for $f'(x)$ directly, we get:

$$f'(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

where

$$\frac{A_0}{2} = \frac{1}{2}\frac{2}{L}\int_0^L f'(x)\,dx = \frac{1}{L}\int_0^L f'(x)\,dx = \frac{1}{L} f(x)\Big|_0^L = \frac{1}{L}(f(L) - f(0))$$

Therefore, if $L = 1$ and $f(x) = x$, this becomes: $A_0/2 = 1$ (as desired). For the other values of n,

$$A_n = \frac{2}{L}\int_0^L f'(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \quad \Rightarrow \quad \begin{array}{ccc} + & \cos(n\pi x/L) & f'(x) \\ - & -(n\pi x/L)\sin(n\pi x/L) & f(x) \end{array}$$

After integration by parts,

$$A_n = \frac{2}{L}\left[\left(f(x)\cos\left(\frac{n\pi x}{L}\right)\right|_0^L + \frac{n\pi}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx\right]$$

The first term is:

$$\frac{2}{L}\left(f(L)\cos(n\pi) - f(0)\right) = \frac{2}{L}(f(L)(-1)^n - f(0))$$

and the second term we can write in terms of B_n, since $B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$. Putting these together, we get our final result:

$$A_n = \frac{2}{L}(f(L)(-1)^n - f(0)) + \frac{n\pi}{L}B_n$$

Comparing the two sets of terms, the derivative is the result of term-by-term differentiation if

$$A_0 = 0 \qquad A_n = \frac{n\pi}{L}B_n$$

This occurs when $f(0) = f(L) = 0$, in which case the Fourier sine series is continuous (if f is continuous on $[0, L]$).

SUMMARY: If the Fourier sine series is continuous, then it is valid to differentiate the sine series term by term.

What about the cosine series?

What happens if $f(x)$ has a cosine series? Let's go through the same arguments as before and see what happens. Take

$$f(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \qquad \Rightarrow \quad A_n = \frac{2}{L}\int_0^L f(x)\cos(n\pi x/L)\,dx$$

Again compare the two forms for the derivative- One by differentiating term by term, the other by looking at the sine series for $f'(x)$.

- Differentiating term-by-term, we get:

$$f'(x) \sim -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin\left(\frac{n\pi x}{L}\right)$$

- Looking at the Fourier sine series of $f'(x)$, we get:

$$f'(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad \Rightarrow \quad B_n = \frac{2}{L}\int_0^L f'(x)\sin(n\pi x/L)\,dx$$

Now, if term-by-term differentiation is valid, we should see that $B_n = -\frac{n\pi}{L}A_n$

Now we work it out by using integration by parts for the integral representation of B_n:

$$B_n = \frac{2}{L}\int_0^L f'(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \quad \Rightarrow \quad \begin{array}{ccc} + & \sin(n\pi x/L) & f'(x) \\ - & (n\pi/L)\cos(n\pi x/L) & f(x) \end{array}$$

Integrating by parts gives us:

$$B_n = \frac{2}{L}\left(f(x)\sin(n\pi x/L)\right)\Big|_0^L - \frac{n\pi}{L}\frac{2}{L}\int_0^L f(x)\cos(n\pi x/L)\,dx = -\frac{n\pi}{L}A_n$$

This seems to always be the case! Let's summarize and expand on what we have found using the next three theorems.

Theorem (Fourier series)

- If f and $f'(x)$ are PWS, and $f(-L) = f(L)$, then the full Fourier series can be differentiated term by term.
- Alternatively, if the Fourier series is continuous and f' is PWS, then the series can be differentiated term by term.

Theorem: Fourier Cosine Series

- If f, f' are PWS on $[0, L]$, then the Fourier cosine series can be differentiated term by term.
- Alternatively, if f' is PWS on $[0, L]$ and the Fourier cosine series is continuous, then the series can be differentiated term by term.

Theorem: Fourier Sine Series

- If f, f' are PWS on $[0, L]$, and $f(0) = F(L) = 0$, then the series can be differentiated term by term.
- Alternatively, if f' is PWS and the Fourier sine series is continuous, then the series can be be differentiated term by term.
- The general formula for the derivative of the FSS: $f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$ is then

$$f'(x) \sim \frac{1}{L}(f(L) - f(0)) + \sum_{n=1}^{\infty}\left[\frac{n\pi}{L}B_n + \frac{2}{L}((-1)^n f(L) - f(0))\right]\cos\left(\frac{n\pi x}{L}\right)$$