Problem Set 8 (4.3, 4.4)

Due: 4.3: 1, 5, 12(a) and 4.2: 1,4

Remember to use our form of the solution:

$$y'' + \omega^2 y = 0 \qquad\qquad y'' - \omega^2 y = 0$$
$$y(t) = C_1\cos(\omega t) + C_2\sin(\omega t) \qquad\qquad y(t) = C_1\cosh(\omega t) + C_2\sinh(\omega t)$$

4.3.1 Solve Laplace's equation:

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \qquad 0 < x < 1, 0 < y < 2 \\ u(x,0) &= 0 \\ u(x,2) &= 10 \\ u(0,y) &= 0 \\ u(1,y) &= 0 \end{aligned}$$

SOLUTION: Separate variables and get the eigenfunctions. Because we have $X(0) = 0$ and $X(1) = 0$, we'll take the eigenfunctions in X.

$$X''Y + Y''X = 0 \quad \Rightarrow \quad \frac{X''Y + Y''X}{XY} = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

Therefore,

$$\begin{array}{c} X'' + \lambda X = 0 \\ X(0) = X(1) = 0 \end{array} \qquad Y'' - \lambda Y = 0$$

For the BVP in X, we know that $\lambda_n = n^2\pi^2$ and $X_n(x) = \sin(n\pi x)$.

Now that we have λ, solve for Y:

$$Y'' - n^2\pi^2 Y = 0 \quad \Rightarrow \quad Y_n(y) = c_n\cosh(n\pi y) + d_n\sinh(n\pi y)$$

Putting our solution together,

$$u(x,y) = \sum_{n=1}^{\infty} \sin(n\pi x)\left[c_n\cosh(n\pi y) + d_n\sinh(n\pi y)\right]$$

That takes care of everything except for the initial condition. Recall that $\cosh(0) = 1$ and $\sinh(0) = 0$ so that $Y_n(0) = c_n$:

$$u(x,0) = 0 = \sum_{n=1}^{\infty} c_n\sin(nx) \quad \Rightarrow \quad c_n = 0$$

That simplifies our solution so far to:

$$u(x,y) = \sum_{n=1}^{\infty} d_n\sin(n\pi x)\sinh(n\pi y)$$

The last boundary condition will define d_n: $u(x, 2) = 10$, so expand 10 in a Fourier sine series in x (with $0 < x < 1$).

$$d_n \sinh(2n\pi) = \frac{2}{1}\int_0^1 10\sin(n\pi x)\,dx = \left(\frac{-20}{\pi n}\cos(n\pi x)\right|_0^1 = -\frac{20}{\pi n}((-1)^n - 1) = \begin{cases} 40/n\pi \text{ if } n \text{ odd} \\ 0 \text{ if } n \text{ even} \end{cases}$$

Don't forget to divide by the hyperbolic sine! Using only the odd indices then $n = 2k - 1$, we get

$$u(x, y) = \frac{40}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)\sinh(2n\pi)}\sin((2k-1)\pi x)\sinh(n\pi y)$$

4.3.5 Solve Laplace's equation:

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \qquad 0 < x < 1, 0 < y < 2 \\ u(x, 0) &= 0 \\ u(x, 2) &= 0 \\ u(0, y) &= y \\ u(1, y) &= 2y \end{aligned}$$

SOLUTION: Separate variables and get the eigenfunctions. Because we have $Y(0) = 0$ and $Y(2) = 0$, we'll take the eigenfunctions in Y.

$$X''Y + Y''X = 0 \quad \Rightarrow \quad \frac{X''Y + Y''X}{XY} = 0 \quad \Rightarrow \quad \frac{Y''}{Y} = -\frac{X''}{X} = -\lambda$$

Therefore,

$$\begin{array}{c} Y'' + \lambda Y = 0 \\ Y(0) = Y(2) = 0 \end{array} \qquad X'' - \lambda X = 0$$

For the BVP in Y, we know that $\lambda_n = \frac{n^2\pi^2}{4}$ and $Y_n(x) = \sin\left(\frac{n\pi}{2}y\right)$.

Now that we have λ, solve for X:

$$X'' - \frac{n^2\pi^2}{4}X = 0 \quad \Rightarrow \quad X_n(x) = c_n\cosh\left(\frac{n\pi}{2}x\right) + d_n\sinh\left(\frac{n\pi}{2}x\right)$$

Putting our solution together,

$$u(x, y) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{2}y\right)\left[c_n\cosh\left(\frac{n\pi}{2}x\right) + d_n\sinh\left(\frac{n\pi}{2}x\right)\right]$$

That takes care of everything except for $u(0, y) = y$ and $u(2, y) = u(2, y)$. These are similar formulas:

$$u(0, y) = \sum_{n=1}^{\infty} c_n\sin\left(\frac{n\pi y}{2}\right) = y \quad \Rightarrow \quad c_n = \frac{2}{2}\int_0^2 y\sin\left(\frac{n\pi y}{2}\right)\,dy$$

We'll compute that at the end, but consider c_n as "computed". Continuing:

$$u(1,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}y\right)\left[c_n \cosh\left(\frac{n\pi}{2}\right) + d_n \sinh\left(\frac{n\pi}{2}\right)\right] = \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi}{2}y\right)$$

where

$$K_n = \frac{2}{2}\int_0^2 2y \sin\left(\frac{n\pi y}{2}\right)\, dy, \qquad \text{so that } K_n = 2c_n$$

Continuing to solve for d_n, we have

$$K_n = 2c_n = c_n \cosh\left(\frac{n\pi}{2}\right) + d_n \sinh\left(\frac{n\pi}{2}\right)$$

Therefore,

$$d_n = \frac{2 - \cosh(n\pi/2)}{\sinh(n\pi/2)} c_n$$

Lastly, computing c_n uses integration by parts:

$$c_n = \frac{4}{n\pi}(-1)^{n+1}$$

4.3.12(a) We want to consider how to use the solutions to the previous two PDEs,

$u_{xx}+u_{yy} = 0, \quad 0<x<1, 0<y<2$	$u_{xx}+u_{yy} = 0, \quad 0<x<1, 0<y<2$
$u(x,0) = 0$	$u(x,0) = 0$
$u(x,2) = 10$	$u(x,2) = 0$
$u(0,y) = 0$	$u(0,y) = y$
$u(1,y) = 0$	$u(1,y) = 2y$

To solve the PDE:

$$\begin{aligned} u_{xx}+u_{yy} &= 0, \qquad 0<x<1, 0<y<2 \\ u(x,0) &= 0 \\ u(x,2) &= 10 \\ u(0,y) &= y \\ u(1,y) &= 2y \end{aligned}$$

SOLUTION: If u_1 solves the upper left PDE and u_2 solves the upper right PDE, then the overall solution is the sum:

$$u(x,y) = u_1(x,y) + u_2(x,y)$$

We note that both u_1, u_2 solve the homogeneous PDE $u_{xx}+u_{yy} = 0$, so by the superposition principle, so does u_1+u_2. The only thing left is to check that the sum satisfies the boundary conditions:

$$\begin{aligned} u(x,0) &= u_1(x,0) + u_2(x,0) &&= 0+0 &&= 0 \\ u(x,2) &= u_1(x,2) + u_2(x,2) &&= 10+0 &&= 10 \\ u(0,y) &= u_1(0,y) + u_2(0,y) &&= 0+y &&= y \\ u(1,y) &= u_1(1,y) + u_2(1,y) &&= 0+2y &&= 2y \end{aligned}$$

Therefore, the sum of the solutions satisfies all boundary conditions.

4.4

4.4.1 In this case, we don't want to solve the PDE, we just want to "convert" the PDE into one with homogeneous BCs.

(a)

$$\begin{aligned} u_{tt} &= u_{xx}, \qquad 0 < x < L, t > 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \\ u(0,t) &= T \\ u(L,t) &= T \end{aligned}$$

In this case, let $u(x,t) = w(x,t) + T$. Since $u_{tt} = w_{tt}$ and $u_{xx} = w_{xx}$, then w would solve the wave equation. Note that $w(x,t) = u(x,t) - T$, so the boundary conditions on w:

- $w(x,0) = u(x,0) - T = f(x) - T$
- $w_t(x,0) = u_t(x,0) - 0 = g(x)$
- $w(0,t) = u(0,t) - T = T - T = 0$
- $w(L,t) = u(L,t) - T = T - T = 0$

(b)

$$\begin{aligned} u_{tt} &= u_{xx}, \qquad 0 < x < L, t > 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \\ u(0,t) &= T \\ u_x(L,t) &= a \end{aligned}$$

In this case, the helper function should satisfy the conditions $v(0) = T$ and $v'(0) = a$. This is a line: $v(x) = ax + T$.

Now let $u(x,t) = w(x,t) + ax + T$. Since $u_{tt} = w_{tt}$ and $u_{xx} = w_{xx}$, then w would solve the wave equation. Note that $w(x,t) = u(x,t) - ax - T$, so the boundary conditions on w:

- $w(x,0) = u(x,0) - ax - T = f(x) - ax - T$
- $w_t(x,0) = u_t(x,0) - 0 = g(x)$
- $w(0,t) = u(0,t) - a(0) - T = T - T = 0$
- $w_x(L,t) = u_x(L,t) - a = a - a = 0$

(c)

$$\begin{aligned} u_{tt} &= u_{xx}, \qquad 0 < x < L, t > 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \\ u_x(0,t) &= a \\ u(L,t) &= T \end{aligned}$$

In this case, the helper function should satisfy the conditions $v'(0) = a$ and $v(L) = T$. This is a line: $v(x) = ax + (T - aL)$.

Now let $u(x,t) = w(x,t) + ax + (T - aL)$. Since $u_{tt} = w_{tt}$ and $u_{xx} = w_{xx}$, then w would solve the wave equation. Note that $w(x,t) = u(x,t) - ax - T + aL$, so the boundary conditions on w:

- $w(x,0) = u(x,0) - ax - T + aL = f(x) - ax - T + aL$
- $w_t(x,0) = u_t(x,0) - 0 = g(x)$
- $w_x(0,t) = u(0,t) - a = a - a = 0$
- $w(L,t) = u(L,t) - ax - T + aL = T - aL - T + aL = 0$

(d)

$$\begin{aligned} u_{tt} &= u_{xx}, \qquad 0 < x < L, t > 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \\ u_x(0,t) &= a \\ u_x(L,t) &= a \end{aligned}$$

In this case, $v'(0) = v'(L) = a$, so we can just make $v(x) = ax$.

Let $u(x,t) = w(x,t) + ax$. Since $u_{tt} = w_{tt}$ and $u_{xx} = w_{xx}$, then w would solve the wave equation. Note that $w(x,t) = u(x,t) - ax$, so the boundary conditions on w:

- $w(x,0) = u(x,0) - ax = f(x) - ax$
- $w_t(x,0) = u_t(x,0) - 0 = g(x)$
- $w_x(0,t) = u_x(0,t) - a = a - a = 0$
- $w_x(L,t) = u_x(L,t) - a = a - a = 0$

4.4.4 Solve the nonhomogeneou heat equation below:

$$\begin{aligned} u_t &= u_{xx} + x, \qquad 0 < x < \pi, t > 0 \\ u(x,0) &= \sin(2x) \\ u(0,t) &= 0 \\ u(\pi,t) &= 0 \end{aligned}$$

SOLUTION: Because we have $X(0) = 0$ and $X(\pi) = 0$, the eigenfunctions will be in X:

$$\lambda_n = n^2 \qquad X_n(x) = \sin(nx)$$

Now we write the solution as:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin(nx)$$

We also have $F(x,t) = x$, or

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t) \sin(nx) \qquad \text{where } F_n(t) = \frac{2}{\pi} \int_0^{\pi} F(x,t) \sin(nx)\, dx$$

In our case, we have no time dependence, so

$$x = \sum_{n=1}^{\infty} F_n \sin(nx) \quad \Rightarrow \quad F_n = \frac{2}{\pi}\int_0^{\pi} x \sin(nx)\, dx$$

so that $F_n = \frac{2}{n}(-1)^{n+1}$. Putting these expressions back into the heat equation, we get

$$\sum_{n=1}^{\infty} b_n'(t)\sin(nx) = \sum_{n=1}^{\infty} -n^2 b_n(t)\sin(nx) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2}{n}\sin(nx)$$

Before we write out the ODEs, let's use the initial condition:

$$u(x,0) = \sin(2x) = \sum_{n=1}^{\infty} b_n(0)\sin(nx)$$

Therefore, $b_n(0) = 0$ except for $b_2(0) = 1$. Now, the ODEs:

For $n \neq 2$, we have

$$b_n'(t) = -n^2 b_n(t) + \frac{(-1)^{n+1}2}{n}, \quad b_n(0) = 0$$

As in class, given $y' = -ky + b$ with $y_0 = 0$, then

$$y(t) = \frac{b}{k}\left(1 - \mathrm{e}^{-kt}\right)$$

Now backsubstituting with $b = (-1)^{n+1}2/n$ and $k = -n^2$, we get

$$b_n(t) = \frac{(-1)^{n+1}2}{n^3}\left(1 - \mathrm{e}^{-n^2 t}\right)$$

In the special case $n = 2$, we have

$$b_n' = -4b_n - 1, \quad b_2(0) = 1 \quad \Rightarrow \quad b_2(t) = \frac{5}{4}\mathrm{e}^{-4t} - \frac{1}{4} = -\frac{1}{4}\left(1 - \mathrm{e}^{-4t}\right) + \mathrm{e}^{-4t}$$

With this, we can now write our solution as:

$$u(x,t) = \mathrm{e}^{-4t}\sin(2x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2}{n^3}\left(1 - \mathrm{e}^{-n^2 t}\right)\sin(nx)$$