Exam 1 Review, Math 367, Spring 2023

- **General notes on the exam:**

  The exam will be about 50 minutes in length, and will cover the material in chapters 1 and 2. You may not use a calculator, but you **may prepare notes**- You can use a half sheet of paper with writing on one side (or the equivalent in area, 8.5" x 5.5").

- **Review of ODEs:**

  Solve a linear first order DE, solve a first order separable DE, solve a second order linear homogeneous DE with constant coefficients, "simple" method of undetermined coefficients. Define $\sinh(x), cosh(x)$, and their basic properties: even/odd, sketch of graphs, derivatives of each.

- **Remark:**

  In section 1.7, the author introduced the *Cauchy-Euler* equation, $t^2y'' + aty' + by = 0$. We won't be using this equation for now, but we will use it later.

- **Know the definitions:**

  PDE, order, solution, initial condition, boundary condition, well-posed, linear operator, homogeneous PDE, linear combination, product solution, eigenvalue (and eigenfunction); Hyperbolic, Parabolic, or Elliptic PDEs; three kinds of BCs- Dirichlet, Neumann, Robin.

- **Other tasks:**

  Be able to verify that a given function is a solution, be able to construct the operator for a given PDE, be able to show that a given operator is linear (or not linear),

- **Modeling:** Be able to identify the "big three" equations. Discuss the three physical "laws" that are used in the construction of the heat equation.

- **Theorems:**

  (1.1) Principle of superposition, structure of solutions to a linear PDE (Exercise 7, Section 1.5),

- **Techniques:**

  Separation of variables, "Separate the PDE", understand by way of example that not all PDEs can have variables separated,

  ("Separate the PDE" means to use a product solution to get a system of ODEs (don't actually solve the ODEs))

Questions

NOTE: I've tried to give you a variety of questions to answer. There are many more problems in the review here than will be on the exam (the number of questions will be reduced so that they can be answered within approximately 50 minutes).

You might have your notes already made before you start the questions, so you can practice looking things up as you go.

1. - Define "eigenvalue" (and eigenfunction) in the context of a linear operator L.
   - Given the definition, how is $y'' + \lambda y = 0$ an eigenvalue problem?
2. What does it mean for a PDE to be **well-posed**?
3. Solve by using ODE methods: $u_{xy} = x - y$
4. Solve by using ODE methods: $u_y + xu = 2$
5. Find all solutions to $u_y = 2x$ (using ODE methods) that also satisfies $u(x, 3) = \sin(x)$.
6. Suppose we're looking for product solutions to $u_x + u_y = 0$. Then we set $u = XY$, and with appropriate algebra, we get
$$\frac{X'}{X} = -\frac{Y'}{Y}$$
What is the justification in setting these equal to a constant?
7. Suppose we are asked to solve the eigenvalue problem
$$\begin{array}{l} y'' + \lambda y = 0 \\ y(a) = y(b) = 0 \end{array}$$
   (a) Suppose we want to change the variable from $a < x < b$ to $z = \frac{L}{b-a}(x - a)$.
      i. Justify: $\dfrac{dy}{dx} = \dfrac{dy}{dz}\dfrac{dz}{dx}$ (and compute what this is given our change of variables).
      ii. Write the BVP in terms of y, z:

   (b) Notice that we changed the interval so that $0 < z < L$. Use this technique to solve $y'' + \lambda y = 0$ with $y(-1) = y(1) = 0$.
8. Is the problem well posed? (You may decide in terms of existence and uniqueness only):
$$y'' + y' - 2y = 0, \quad y(0) = 0, y'(1) = 0, 0 \le x \le 1$$
9. For each operator below, either prove that it is linear, or show that it is not linear:
   (a) $L(u) = yu_x - x^2u_y + 2u$
   (b) $L(u) = u - u_{xxt} + uu_{tt}$

10. Suppose that u_1 solves $u_t - u_{xx} = f(x,t)$ and u_2 solves $u_t - u_{xx} = g(x,t)$ for some functions f, g. Find a solution u_3 that will solve the equation $u_t - u_{xx} = 5f(x,t) - 7g(x,t)$.

11. Classify the heat equation, the wave equation and Laplace's equation as **hyperbolic**, **parabolic**, or **elliptic** using the definition (that is, show your work).

12. Use ODE techniques to find the general solution of the following, where $u = u(x,y)$:

$$yu_{xy} + 2u_x = x$$

Hint: The equation can be expressed first as $(...)_x = x$.

13. Given your solution to the previous problem, find a particular solution to the boundary conditions:

$$u(x,1) = 0 \qquad u(0,y) = 0$$

14. Solve the following PDE using separation of variables. You do not need to justify your solutions to the underlying eigenvalue problem.

$$\begin{array}{l} u_t = u_{xx} \quad 0 < x < 2, t > 0 \\ u(0,t) = 0 \\ u(2,t) = 0 \\ u(x,0) = 3\sin(\pi x) - 4\sin((3\pi/2)x), 0 < x < 2 \end{array}$$

15. Separate the PDE into a system of ODEs (you do NOT need to solve each ODE, just set them up).

    (a) $u_{xx} - xu_y + xu = 0$

    (b) $u_t = u_{xx} + u_{yy}$

16. Solve the PDE by using separation of variables: $y^2u_x + x^2u_y = 0$

17. Solve the following eigenvalue problem: $y'' + 2y' + (\lambda + 1)y = 0$, with $y(0) = 0$ and $y(\pi) = 0$.

18. (a) Given the eigenvalue problem $y'' + \lambda y = 0$ with $y(0) = 0$ and $y(L) = 0$, and if we are told that λ_1, λ_2 are two distinct eigenvalue with associated eigenfunctions y_1, y_2, then show that

$$(\lambda_1 - \lambda_2)\int_0^L y_1y_2\,dx = \int_0^L (y_1y_2'' - y_1''y_2)\,dx$$

    (b) Use integration by parts to show

$$\int_0^L y_1y_2''\,dx = y_1y_2'|_0^L - \int_0^L y_1'y_2'\,dx$$

19. The **steady state** or **equilibrium** solution to the heat equation is a solution that does not depend on time (that is, $u = u(x)$).

    (a) Find all the steady state solutions to the heat equation if the left end of the rod is held at 3 degrees, and the right end is held at 10 degrees.

    (b) Repeat the previous problem, but the left end is insulated and the right end is held at 10 degrees.

20. Given the heat equation with nonhomogeneous boundary conditions:

$$\begin{aligned} &u_t = u_{xx} \\ &u(x,0) = f(x) \\ &u(0,t) = 10 \\ &u(5,t) = 30 \end{aligned}$$

    (a) Find an equilibrium solution (call it $v(x)$ rather than $u(x)$) that satisfies the nonhomogeneous boundary conditions.

    (b) Define $w(x,t) = u(x,t) - v(x)$. Write the heat equation in terms of w rather than u. Verify that w solves the heat equation with homogeneous boundary conditions.

    (c) Remark: Using this, the overall original solution is $u(x,t) = w(x,t) + v(x)$.

21. Solve the heat equation intial-boundary-value problem

$$\begin{aligned} &u_t = u_{xx} \\ &u(x,0) = 3 + \cos(2\pi x) \\ &u_x(0,t) = 0 \\ &u_x(3,t) = 0 \end{aligned}$$

    You do not need to justify your solutions to the underlying eigenvalue problem.

22. Solve the wave equation intial-boundary-value problem

$$\begin{aligned} &u_{tt} = u_{xx} \\ &u(x,0) = 5\sin(2x) - 7\sin(4x) \\ &u_t(x,0) = 0 \\ &u(0,t) = u(\pi,t) = 0 \end{aligned}$$

    You do not need to justify your solutions to the underlying eigenvalue problem.

23. Solve Laplace's equation:

$$\begin{aligned} &u_{xx} + u_{yy} = 0 \\ &u(x,0) = \sin(3x) \\ &u(x,1) = \sin(x) \\ &u(0,y) = u(\pi,y) = 0 \end{aligned}$$

    You do not need to justify your solutions to the underlying eigenvalue problem.