Solutions to the Review, Exam 3, Math 367

(Corrected copy, Apr 20)

1. Solve, using a suitable change of coordinates: $u_t + xu_x = 1$ with $u(x,0) = f(x)$.

   SOLUTION:
$$\frac{dt}{1} = \frac{dx}{x} \quad \Rightarrow \quad dt = \frac{1}{x}\,dx \quad \Rightarrow \quad \int dt = \int \frac{1}{x}\,dx$$
   so we get $t + C = \ln(x)$, or $C = \ln(x) - t$. This gives the change of coordinates:
$$\begin{array}{l} \xi = x \\ \eta = \ln(x) - t \end{array} \quad \Rightarrow \quad \begin{array}{l} u_x = u_\xi \cdot 1 + u_\eta \frac{1}{x} \\ u_t = u_\xi \cdot 0 + u_\eta(-1) \end{array} \quad \Rightarrow \quad u_t + xu_x = xu_\xi = \xi u_\xi$$
   Now the PDE becomes
$$\xi u_\xi = 1 \quad \Rightarrow \quad u(\xi, \eta) = \ln(\xi) + g(\eta)$$
   (where g is arbitrary). Backsubstituting,
$$u(x,t) = \ln(x) + g(\ln(x) - t)$$
   Using the initial conditions to satisfy the initial condition:
$$u(x,0) = \ln(x) + g(\ln(x)) = f(x) \quad \Rightarrow \quad g(\ln(x)) = f(x) - \ln(x)$$
   We still want $g(z)$, so let $z = \ln(x)$, or $x = \mathrm{e}^z$. Therefore,
$$g(z) = f(\mathrm{e}^z) - z$$
   Now we can write the full solution as:
$$u(x,t) = \ln(x) + g(\ln(x) - t) = \ln(x) + f(\mathrm{e}^{\ln(x)-t}) - (\ln(x) - t)$$
   The solution simplifies to
$$u(x,t) = f(x\mathrm{e}^{-t}) + t$$

2. Solve the wave equation using D'Alembert's equation:

   (a)
$$\begin{array}{l} u_{tt} = u_{xx}, \quad -\infty < x < \infty, t > 0 \\ u(x,0) = x \\ u_t(x,0) = 1 \end{array}$$
   SOLUTION: Using the formula
$$\frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$
   the solution here is:
$$\frac{1}{2}\left[x + ct + x - ct\right] + \frac{1}{2}\int_{x-t}^{x+1} 1\,dz = x + (x+t) - (x-t) = x + t$$

(b)

$$\begin{aligned}&u_{tt} = 4u_{xx}, \quad -\infty < x < \infty, t > 0\\ &u(x,0) = 0\\ &u_t(x,0) = \cos(x)\end{aligned}$$

SOLUTION: Using the formula

$$\frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

the solution here is:

$$\frac{1}{4}\int_{x-2t}^{x+2t} \cos(z)\,dz = \frac{1}{4}\sin(z)\Big|_{x-2t}^{x+2t} = \frac{1}{4}\left[\sin(x+2t) - \sin(x-2t)\right]$$

3. We'll solve the same wave equations as the previous problem, but now the interval is $x \in [0, L]$ and use the boundary conditions: $u(0,t) = u(L,t) = 0$. We'll solve this "formally", meaning that you should write down any integrals you would need to evaluate, but you may leave them unevaluated.

(a)

$$\begin{aligned}&u_{tt} = u_{xx}, \quad 0 < x < L, t > 0\\ &u(x,0) = x\\ &u_t(x,0) = 1\\ &u(0,t) = 0\\ &u(L,t) = 0\end{aligned}$$

SOLUTION: Separating variables,

$$\begin{array}{ll} X'' + \lambda X = 0 & \\ X(0) = X(L) = 0 & T'' + \lambda_n T = 0 \\ \lambda_n = (n\pi/L)^2 & T_n(t) = c_n\cos(n\pi t/L) + d_n \sin(n\pi t/L) \\ X_n(x) = \sin(n\pi x/L) & \end{array}$$

Putting the solution together, we have

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x/L)\left[c_n\cos(n\pi t/L) + d_n\sin(n\pi t/L)\right]$$

To find c_n, d_n, we use the initial conditions- For $u(x,0)$, we'll have

$$x = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L) \quad \Rightarrow \quad c_n = \frac{2}{L}\int_0^L x\sin(n\pi x/L)\,dx$$

(We can leave c_n like that since we're looking for a "formal" solution)

Now for d_n, we compute $u_t(x,t)$ first:

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x/L)\left[-(n\pi/L)c_n \sin(n\pi t/L) + (n\pi/L)d_n \cos(n\pi t/L)\right]$$

Now use $u_t(x,0) = 1$ to get

$$1 = \sum_{n=1}^{\infty}(-n\pi/L)d_n \sin(n\pi x/L) \quad \Rightarrow \quad d_n = \frac{2}{n\pi}\int_0^L \sin(n\pi x/L)\,dx$$

Now we have integral expressions for the coefficients.

(b)

$$\begin{array}{l} u_{tt} = 4u_{xx}, \quad 0 < x < L, t > 0 \\ u(x,0) = 0 \\ u_t(x,0) = \cos(x) \\ u(0,t) = 0 \\ u(L,t) = 0 \end{array}$$

SOLUTION: Separating variables,

$$\begin{array}{ll} X'' + \lambda X = 0 & \\ X(0) = X(L) = 0 & T'' + 4\lambda_n T = 0 \\ \lambda_n = (n\pi/L)^2 & T_n(t) = c_n \cos(2n\pi t/L) + d_n \sin(2n\pi t/L) \\ X_n(x) = \sin(n\pi x/L) & \end{array}$$

Putting the solution together, we have

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x/L)\left[c_n \cos(2n\pi t/L) + d_n \sin(2n\pi t/L)\right]$$

To find c_n, d_n, we use the initial conditions- For $u(x,0) = 0$, we'll have

$$0 = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L) \quad \Rightarrow \quad c_n = 0$$

Now for d_n, we compute $u_t(x,t)$ first:

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x/L)(2n\pi/L)d_n \cos(2n\pi t/L)$$

Now use $u_t(x,0) = \cos(x)$ to get

$$\cos(x) = \sum_{n=1}^{\infty}(2n\pi/L)d_n \sin(n\pi x/L) \quad \Rightarrow \quad d_n = \frac{1}{n\pi}\int_0^L \cos(x)\sin(n\pi x/L)\,dx$$

4. Consider

$$\begin{aligned}
&u_t = u_{xx} \qquad 0 < x < 1\\
&u(x,0) = f(x)\\
&u_x(0,t) = t\\
&u_x(1,t) = t^2
\end{aligned}$$

Convert the PDE into one with homogeneous boundary conditions, and write the new PDE that one would need to solve (with the new BCs).

SOLUTION: We also did this one in class. The idea is that we're treating t and t^2 as if they were constants. For example, we could set $t = A$ and $t^2 = B$, then find the simplest form for $v'(x)$ so that

$$v'(0) = A \text{ and } v'(1) = B$$

The line between $(0, A)$ and $(1, B)$ is (in point-slope form, using $(0, A)$):

$$v'(x) - A = \frac{B-A}{1}(x-0) \quad \Rightarrow \quad v'(x) = (t^2 - t)x + t$$

Therefore,

$$v(x,t) = \frac{1}{2}x^2(t^2 - t) + xt$$

Now, we set $u(x,t) = w(x,t) + v(x,t)$ and find the PDE that w will satisfy (which will include the homogeneous BCs)- Remember that u solves the original PDE.

$$u_t = w_t + v_t = w_t + x^2 t - \frac{1}{2}x^2 + x$$

and

$$u_{xx} = w_{xx} + (t^2 - t)$$

Therefore, if $u_t = u_{xx}$, then for w we have

$$w_t = w_{xx} + (t^2 - t) - x^2 t + \frac{1}{2}x^2 - x$$

For the initial and boundary conditions:

$$\begin{aligned}
&w(x,0) = u(x,0) - v(x,0) = f(x)\\
&w_x(0,t) = 0\\
&w_x(1,t) = 0
\end{aligned}$$

5. Solve the nonhomogeneous PDE below.

$$\begin{aligned}
&u_{tt} = u_{xx} + \sin(x) \qquad 0 < x < \pi\\
&u(x,0) = \sin(3x)\\
&u_t(x,0) = \sin(5x)\\
&u(0,t) = 0\\
&u(\pi,t) = 0
\end{aligned}$$

Remember to use the eigenfunction approach, which in this case will be $X_n(x) = \sin(nx)$. We write u using a generic $T_n(t)$:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(nx)$$

We'll be putting this back into the PDE, but we can get some initial conditions first.

$$u(x,0) = \sin(3x) = \sum_{n=1}^{\infty} T_n(0)\sin(nx) \quad \Rightarrow \quad T_n(0) = \begin{cases} 0 & \text{if } n \neq 3 \\ 1 & \text{if } n = 3 \end{cases}$$

$$u_t(x,0) = \sin(5x) = \sum_{n=1}^{\infty} T_n'(0)\sin(nx) \quad \Rightarrow \quad T_n'(0) = \begin{cases} 0 & \text{if } n \neq 5 \\ 1 & \text{if } n = 5 \end{cases}$$

Now put $u(x,t)$ into the PDE:

$$\sum_{n=1}^{\infty} T_n''(t)\sin(nx) = \sum_{n=1}^{\infty} -n^2 T_n(t)\sin(nx) + \sin(x)$$

We'll have three special cases, then all the other cases:

- $n = 1$ (because of the $\sin(x)$ on the right side):

$$T_1'' + T = 1 \quad T_1(0) = 0, T_1'(0) = 0 \quad \Rightarrow \quad T_1(t) = 1 - \cos(t)$$

- $n = 3$:

$$T_3'' + 9T_3 = 0 \quad T_3(0) = 1, T_3'(0) = 0 \quad \Rightarrow \quad T_3(t) = \cos(3t)$$

- $n = 5$:

$$T_5'' + 25T_5 = 0 \quad T_5(0) = 0, T_5'(0) = 1 \quad \Rightarrow \quad T_5(t) = \frac{1}{5}\sin(5t)$$

- All other n's:

$$T_n'' + n^2 T_n = 0 \text{ with } T_n(0) = 0, T_n'(0) = 0 \quad \Rightarrow \quad T_n(t) = 0$$

Now put it all together:

$$u(x,t) = \sin(x)(1 - \cos(t)) + \sin(3x)\cos(3t) + \frac{1}{5}\sin(5x)\sin(5t)$$

6. In D'Alembert's solution to the wave equation, we started with the equation below and made the following change of coordinates:

$$u_{tt} = c^2 u_{xx} \qquad \begin{aligned} \xi &= x + ct \\ \eta &= x - ct \end{aligned}$$

Write down the new PDE we get in $u(\xi, \eta)$. Show your work!

SOLUTION:

$$\begin{aligned} u_t &= u_\xi \xi_t + u_\eta \eta_t = c u_\xi - c u_\eta \\ u_{tt} &= (c u_\xi - c u_\eta)_\xi \xi_t + (c u_\xi - c u_\eta)_\eta \eta_t = c^2(u_{\xi\xi} - 2u_{\eta\xi} + u_{\eta\eta}) \end{aligned}$$

Similarly,

$$\begin{aligned} u_x &= u_\xi + u_\eta \\ u_{xx} &= u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta} \end{aligned}$$

Therefore,

$$c^2 u_{xx} - u_{tt} = 4c^2 u_{\eta\xi} = 0 \quad \Rightarrow \quad u_{\eta\xi} = 0$$

7. Given $x^2 u_x + y u_y + xyu = 1$, use a change of coordinates so that the new equation involves only one derivative, and write the new equation (using variables ξ, η). Do NOT solve the PDE.

SOLUTION:

$$\frac{dx}{x^2} = \frac{dy}{y}$$

The variables are already separated, so integrate both sides.

$$\int x^{-2}\, dx = \int \frac{1}{y}\, dy \quad \Rightarrow \quad C = \ln(y) + x^{-1}$$

That is our change of variables:

$$\begin{aligned} \xi &= x \\ \eta &= \ln(y) + x^{-1} \end{aligned}$$

Therefore, $x^2 u_x + y u_y = x^2 u_\xi = \xi^2 u_\xi$, and the PDE so far is:

$$\xi^2 u_\xi + xyu = 1$$

For the last substitution, $x = \xi$, and $y = \mathrm{e}^{\eta - 1/\xi}$, so our PDE becomes:

$$\xi^2 u_\xi + \xi \mathrm{e}^{\eta - 1/\xi} u = 1$$

8. Solve the heat equation $u_t = 4u_{xx}$ for a rod of length π with both ends insulated if $u(x, 0) = f(x)$. You may formally solve the PDE, meaning any integrals should be written down, but you can leave them unevaluated.

SOLUTION: "Both ends insulated" means that $u_x(0, t) = u_x(\pi, t) = 0$. Therefore, we have the 4th case in the eigenfunctions summary, so that

$$\begin{array}{lcl} \lambda_0 = 0, X_0(x) = 1 & \Rightarrow & T' = 0 & \Rightarrow & T_0(t) = C_0 \\ \lambda_n = n^2, X_n(x) = \cos(nx) & & T' + 4n^2 T = 0 & & T_n(t) = C_n \mathrm{e}^{-4n^2 t} \end{array}$$

Now the solution is given by:

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n \mathrm{e}^{-4n^2 t} \cos(nx)$$

Now use the initial conditions to determine the coefficients:

$$u(x,0) = f(x) = C_0 + \sum_{n=1}^{\infty} C_n \cos(nx)$$

so that

$$C_0 = \frac{a_0}{2} = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx$$

$$C_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx$$

9. Give a formal solution to the wave equation below (meaning write down any integrals you're computing, but you may leave them unevaluated).

$$\begin{aligned} &u_{tt} = 9u_{xx} \\ &u(x,0) = f(x) \\ &u_t(x,0) = g(x) \\ &u(0,t) = 0 \\ &u_x(L,t) = 0 \end{aligned}$$

SOLUTION: Notice that the interval for x is finite; $0 < x < L$, so that we cannot use D'Alembert's solution (the way we derived it). Therefore, we have to separate variables. The BCs lead us to BC2 on the table, so that we get the following (in the last entry, (2L) is in the denominator).

$$\begin{aligned} &X'' + \lambda X = 0 \\ &X(0) = 0 \\ &X'(L) = 0 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} &\lambda_n = ((2n-1)\pi/2L)^2 \\ &X_n(x) = \sin((2n-1)\pi x/2L) \end{aligned}$$

For T, we have

$$\begin{aligned} &T'' + 9\lambda_n T = 0 \\ &T_n(t) = C_n \cos(3\sqrt{\lambda_n} t) + D_n \sin(3\sqrt{\lambda_n} t) \end{aligned}$$

Now for the full sum:

$$u(x,t) = \sum_{n=1}^{\infty} \sin((2n-1)\pi x/2L) \left[C_n \cos(3\sqrt{\lambda_n} t) + D_n \sin(3\sqrt{\lambda_n} t) \right]$$

We should find

$$C_n = \frac{2}{L} \int_0^L f(x) \sin((2n-1)\pi x/2L)\, dx$$

and

$$3\sqrt{\lambda_n} D_n = \frac{2}{L} \int_0^L g(x) \sin((2n-1)\pi x/2L)\, dx$$

(then solve for D_n).