Linear ODE Theory (Sections 3.2 and 3.3)

In the following table, take L to be the linear operator:

$$L(y) = y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_{n-1}(t)y' + p_n(t)y$$

where $y^{(k)}$ denotes the k^{th} derivative of y with respect to t.

In Linear Algebra:	**In Differential Equations:**
If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator, then $T(x) = b$ can be written as $$A\mathbf{x} = \mathbf{b}$$	If $L : C^n[0,1] \to C[0,1]$ is a linear operator, the second order linear diff. equation can be written as: $$L(y) = f(t)$$
The Nullspace of A is important, because all solutions are written in the form: $$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$$ where $\mathbf{x}_h$ is the homogeneous part of the solution, $A\mathbf{x}_h = 0$. $\boldsymbol{x}_p$ is called the particular part of the solution.	The Nullspace of L is important, because all solutions are written in the form: $$y(t) = y_h + y_p$$ where y_h solves: $L(y) = 0$, and is called the homogeneous part of the solution. y_p is called the particular solution.
The dimension of the nullspace tells us how many "pieces" there are to $\mathbf{x}_h$, in the sense that: $$\mathbf{x}_h = c_1\mathbf{v}_1 + \ldots + c_p\boldsymbol{v}_p$$ where the $\mathbf{v}_i$ are linearly independent. (This is the **Principle of Superposition**)	The dimension of the nullspace tells us how many "pieces" there are to y_h, in the sense that: $$y_h = c_1y_1 + \ldots + c_py_p$$ where the y_i are "linearly independent" functions. (This is the **Principle of Superposition**)
	The dimension of the nullspace of L is n, the order of the differential equation. (Proved later).
Definition: $\{\boldsymbol{v}_i\}_{i=1}^p$ are linearly independent iff the only solution to: $$c_1\boldsymbol{v}_1 + \ldots + c_p\boldsymbol{v}_p = 0$$ is $c_1 = c_2 = \ldots = c_n = 0$.	**Definition:** $\{y_1(t), \ldots, y_p(t)\}$ are linearly independent functions iff the only solution to: $$c_1y_1(t) + \ldots + c_py_p(t) = 0$$ is $c_1 = c_2 = \ldots = c_n = 0$. The same constants must work for all time.
Checking for linear independence of n vectors in $\mathbb{R}^n$: Check that $$\det[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n] \neq 0$$	Checking for linear independence of n solutions: Check that the **Wronskian** (at some t_0) is non-zero (this is an $n \times n$ matrix): $$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0$$

We now get more specific. We are interested in solving:

$$ay'' + by' + cy = f(t) \quad \text{or} \quad L(y) = f(t) \tag{1}$$

1. All solutions to Equation (1) are of the form:

$$y(t) = y_h(t) + y_p(t)$$

   where y_h is called the homogeneous part of the solution, and y_p is called the particular part of the solution.

2. The dimension of the nullspace is two (proved in class). Therefore,

$$y_h = c_1 y_1 + c_2 y_2$$

   where y_1 and y_2 are linearly independent solutions to $L(y) = 0$.

   > **Definition:** A set of linearly independent solutions is said to be a **fundamental set** of solutions if every solution to the initial value problem, $L(y) = 0, y(0) = y_0$ is a linear combination of the fundamental set (therefore, a fundamental set is another name for a *basis*).

3. Are y_1 and y_2 linearly independent? From the definition, we would need to know that

$$c_1 y_1 + c_2 y_2 = 0$$

   In the case of two functions, this is easy to check. Two functions are linearly independent iff they are not constant multiples of each other. For the more general case of k functions, we need to compute what's called the *Wronskian*. See your text for more details.

4. Construction of the Fundamental Set, y_1 and y_2:

   How can we construct a fundamental set of solutions? We solve the following initial value problems:

$$ay'' + by' + cy = 0, \quad y(0) = 1, y'(0) = 0 \tag{2}$$
$$ay'' + by' + cy = 0, \quad y(0) = 0, y'(0) = 1 \tag{3}$$

   Let the solution to IVP (2) be $y_1(t)$, and the solution to IVP (3) be $y_2(t)$. Then the initial conditions *guarantee* that y_1 and y_2 are linearly independent, and since we know that there are only two solutions in the fundamental set, we have constructed the entire fundamental set.

   Using this fundamental set, the solution to an arbitrary initial value problem:

$$ay'' + by' + cy = 0, \quad y(0) = y_0, y'(0) = v_0$$

   is:

$$y(t) = y_0 y_1(t) + v_0 y_2(t)$$

5. Putting it all together: Given $ay'' + by' + cy = 0$, we solve by assuming that $y(t) = \mathrm{e}^{rt}$. This leads us to solving the *characteristic equation*:

$$ar^2 + br + c = 0$$

   for r. We solve this by using the quadratic formula. We have seen that we obtain a fundamental set of solutions if we have two real solutions to the characteristic equation.

   The rest of chapter three discusses how we obtain a fundamental set of solutions if we only have one real root, and how to interpret the solutions if we have two complex conjugate roots.