Complex Solutions to Real Solutions and a Computation

1 Sines and Cosines

We begin with a "side computation". Using Euler's Formula, $e^{ibt} = \cos(bt) + i\sin(bt)$, we see that:

$$\begin{aligned}\frac{e^{ibt}+e^{-ibt}}{2} &= \frac{\cos(bt)+i\sin(bt)+\cos(bt)-i\sin(bt)}{2}\\ &= \frac{2\cos(bt)}{2}\\ &= \cos(bt)\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{e^{ibt}-e^{-ibt}}{2i} &= \frac{\cos(bt)+i\sin(bt)-\cos(bt)+i\sin(bt)}{2i}\\ &= \frac{2i\sin(bt)}{2i}\\ &= \sin(bt)\end{aligned}$$

2 The Solution

Now we consider solutions to the second order, linear, homogeneous differential equation:

$$ay'' + by' + cy = 0$$

From the characteristic equation, $ar^2+br+c = 0$, we suppose that the solutions are complex conjugates:

$$r = s \pm iw$$

Note that in this case, while s could be zero, part of the assumption is that $w \neq 0$ (Otherwise, we would have one real solution to the characteristic equation).

Then, as usual, we take $y = c_1 e^{(s+iw)t} + c_2 e^{(s-iw)t}$. We show that a fundamental set of solutions is given by:

$$y_1(t) = e^{st}\cos(wt), \quad y_2(t) = e^{st}\sin(wt)$$

which removes the complex number i (in fact, any solution to an initial value problem will formulate the c_1, c_2 in such a way as to remove the complex number i).

From the general form of the solution, solve the specific initial value problems, with initial values:

$$\begin{array}{ll} y_1(0) = 1 & y_2(0) = 0\\ y_1'(0) = s & y_2'(0) = w\end{array}$$

Then, we see that the Wronskian will be non-zero for any valid values of s, w.

Let's look at y_1 first: If $y_1(0) = 1$, then $c_1 + c_2 = 1$. The second condition gives:

$$\begin{aligned} y_1'(0) &= s \quad \text{so :}\\ s(c_1+c_2) + ib(c_1-c_2) &= s\\ ib(c_1-c_2) &= 0 \quad (c_1+c_2=1)\\ c_1 - c_2 &= 0\end{aligned}$$

From which we get that $c_1 = c_2 = \frac{1}{2}$. Then:

$$y_1(t) = \frac{e^{(s+iw)t}+e^{(s-iw)t}}{2} = \frac{e^{st}(e^{iwt}+e^{-iwt})}{2} = e^{st}\cos(wt)$$

We now consider y_2: If $y_2(0) = 0$, then $c_1 + c_2 = 0$. The second condition gives:

$$\begin{aligned} y_2'(0) &= w \quad \text{so :} \\ s(c_1 + c_2) + iw(c_1 - c_2) &= w \\ iw(c_1 - c_2) &= w \quad (c_1 + c_2 = 0) \\ c_1 - c_2 &= \frac{1}{i} \end{aligned}$$

From which we get that $c_1 = \frac{1}{2i}, c_2 = -\frac{1}{2i}$. Then:

$$y_2(t) = \frac{\mathrm{e}^{(s+iw)t} - \mathrm{e}^{(s-iw)t}}{2i} = \frac{\mathrm{e}^{st}(\mathrm{e}^{iwt} - \mathrm{e}^{-iwt})}{2i} = \mathrm{e}^{st}\sin(wt)$$

We now conclude with the following. If we are given:

$$ay'' + by' + cy = 0$$

with roots to the characteristic equation:

$$ar^2 + br + c = 0$$

complex conjugates, $r = s \pm iw$, then the general solution to the seond order linear homogeneous differential equation is:

$$y_h(t) = \mathrm{e}^{st}\left(c_1\cos(wt) + c_2\sin(wt)\right)$$

(Note the absence of the complex number i).

3 Analysis of the Solution

It is convenient to rewrite:

$$c_1\cos(wt) + c_2\sin(wt)$$

as a single periodic function. This will allow for a quick sketch of the solution, and the analysis will be easier than for the sum. Using the trig identity:

$$R\cos(\omega t - \delta) = R\cos(\delta)\cos(\omega t) + R\sin(\delta)\sin(\omega t)$$

we see that $c_1 = R\cos(\delta), c_2 = R\sin(\delta)$. From this, we get that, to convert:

$$c_1\cos(wt) + c_2\sin(wt) = R\cos(\omega t - \delta)$$

we set $R = \sqrt{c_1^2 + c_2^2}$ and $\delta = \mathrm{Tan}^{-1}(c_2/c_1)$.

Note that the inverse tangent is:

- Taken on the sine coefficient divided by the cosine coefficient.
- δ is the actual angle- that is, from the signs of c_1 and c_2, we need to know which quadrant δ is in, and make the appropriate change.

Re-writing the solution then yields:

$$y(t) = R\mathrm{e}^{st}\cos(\omega t - \delta)$$

The **pseudoperiod** of y is $\frac{2\pi}{\omega}$, and the **phase angle** is δ.