Complex Solutions to Real Solutions, Part II

CONTEXT: We are solving $\boldsymbol{x}' = A\boldsymbol{x}$ and we have obtained a complex eigenvalue, λ. Corresponding to λ is the eigenvector $\boldsymbol{v}$. Why can go from:

$$c_1 e^{\lambda t}\boldsymbol{v} + c_2 e^{\bar{\lambda} t}\bar{\boldsymbol{v}}$$

To:

$$C_1 \mathrm{Re}\left(e^{\lambda t}\boldsymbol{v}\right) + C_2 \mathrm{Im}\left(e^{\lambda t}\boldsymbol{v}\right)$$

To show why, we state a theorem:

Theorem: If a set of eigenvalues is distinct, the eigenvectors are linearly independent. (See Chapter 5.1, Theorem 2 of David Lay's Linear Algebra book, for example).

In particular, this theorem holds for λ, $\bar{\lambda}$. Therefore, $\boldsymbol{v}$ and $\bar{\boldsymbol{v}}$ are linearly independent.

We now show that $\mathrm{Re}(\boldsymbol{v})$ and $\mathrm{Im}(\boldsymbol{v})$ are linearly independent vectors.

Let $\boldsymbol{v} = \boldsymbol{a} + i\boldsymbol{b}$, so that $\bar{\boldsymbol{v}} = \boldsymbol{a} - i\boldsymbol{b}$. Since $\boldsymbol{v}, \bar{\boldsymbol{v}}$ are linearly independent, the only solution to the following equation is trivial:

$$c_1 \left(\boldsymbol{a} + i\boldsymbol{b}\right) + c_2 \left(\boldsymbol{a} - i\boldsymbol{b}\right) = \boldsymbol{0}$$

which is equivalent to saying that the only solution to the following equation is trivial:

$$(c_1 + c_2)\boldsymbol{a} + i(c_1 - c_2)\boldsymbol{b} = \boldsymbol{0}$$

But the solution to this equation is equivalent to the solution of:

$$k_1\boldsymbol{a} + k_2\boldsymbol{b} = \boldsymbol{0}$$

Therefore, we have proven that the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent. Therefore, the real part of $\boldsymbol{v}$ and the imaginary part of $\boldsymbol{v}$ are linearly independent, and:

$$\mathrm{Re}\left(e^{\lambda t}\boldsymbol{v}\right) \text{ and } \mathrm{Im}\left(e^{\lambda t}\boldsymbol{v}\right)$$

are linearly independent solutions.

We might also compare this with what we said in the second order linear differential equation case:

If:

$$k_1 y'' + k_2 y' + k_3 y = 0$$

and $r = a \pm ib$, then the solution is:

$$y(t) = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)$$

which can also be written as:

$$y(t) = c_1 \mathrm{Re}(e^{rt}) + c_2 \mathrm{Im}(e^{rt})$$