Complex Numbers

1 Introduction

1.1 Real or Complex?

As you might recall, the Real Number System is made up of rational numbers (those that can be written as $\frac{p}{q}$, with p and q integers) and irrational numbers (like π, $\sqrt{2}$, etc.).

The Complex Number System is made up of objects that look like:

$$z = a + bi \tag{1}$$

where a, b are Real Numbers and $i = \sqrt{-1}$. Some examples of complex numbers: $5 + 2i$, $3 - \sqrt{2}i$, 4.23 (which is $4.23 + 0i$), etc. Notice that by taking $b = 0$ in the definition in Equation (1), we can get any Real Number. Therefore, the set of Complex Numbers is "bigger" than the set of Real Numbers.

1.2 Visualizing Complex Numbers

A complex number is defined by it's two real numbers. If we have $z = a + bi$, then we say that the *real part* of z is the number "a", and the *imaginary part* of z is the number b (without the i). To visualize a complex number, we can plot it on the plane. The horizontal axis is for the Real part, and the vertical axis is for the Imaginary part. For example, the complex number $3-5i$ is plotted as the ordered pair $(3, -5)$. The complex number i is plotted as $(0, 1)$, and the complex number 5 is plotted as $(5, 0)$.

1.3 Operations on Complex Numbers

1.3.1 The Conjugate of a Complex Number

If $z = a + bi$ is a complex number, then its *conjugate*, denoted by $\bar{z}$ is $a - bi$. For example,

$$z = 3 + 5i \Rightarrow \bar{z} = 3 - 5i$$
$$z = i \Rightarrow \bar{z} = -i$$
$$z = 3 \Rightarrow \bar{z} = 3$$

Graphically, the conjugate of a complex number is it's mirror image across the horizontal axis.

1.3.2 The Size of a Complex Number

The size, or absolute value, of a complex number, denoted by $|z|$, is graphically it's distance to the origin. Therefore, if $z = a+bi$, then it's plotted as (a, b), and it's distance to the origin is $\sqrt{a^2 + b^2}$.

1.3.3 Addition, Subtraction and Multiplication

To add (or subtract) two complex numbers, add (or subtract) the real parts and the imaginary parts separately:

$$(3 - 5i) + (2 + i) = (3 + 2) + (-5 + 1)i = 5 - 4i$$
$$(3) - (2 - 3i) = (3 - 2) + (0 - (-3))i = 1 + 3i$$

To multiply complex numbers (recall that $i^2 = -1$):

$$(a+bi)(c+di) = ac+adi+bci+bdi^2 = (ac-bd)+(ad+bd)i$$

which is just the "FOIL" trick we learn for multiplying things like $(3x - 2)(x + 5)$.

To multiply a real number times a complex number:

$$a(c + di) = ac + adi$$

1.3.4 Division by a Complex Number

Let $z = a + bi$ and $w = c + di$. The we interpret $\frac{z}{w}$ in the following way:

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

In this way, we change division by a complex number into division by a real number ($|w|^2$).

Example:

$$\frac{1 + 2i}{3 - 5i} = \frac{(1 + 2i)(3 + 5i)}{34} = \frac{-7}{34} + \frac{11}{34}i$$

1.4 Alternate Representations of Complex Numbers

We've shown that we can represent a complex number as $a + bi$ or graphically as the ordered pair (a, b). We can also represent a complex number in it's *polar* form. In this case,

$$z = re^{i\theta}$$

where $r = |z|$, and θ is the angle that the point (a, b) makes with the positive real axis.

1.4.1 Interpretation of $e^{i\theta}$

Definition:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

Examples of working with this definition:

$$e^0 = \cos(0) + i \sin(0) = 1$$
$$e^{3+5i} = e^3e^{5i} = e^3 \cos(5) + ie^3 \sin(5)$$
$$e^{\pi i} = \cos(\pi) + i \sin(\pi) = -1$$

From the second example above, we see that the laws of exponents still work when the exponent is complex!

2 Real Polynomials and Complex Numbers

Real polynomials are polynomials with real numbers as coefficients. For example, quadratic polynomials look like:

$$ax^2 + bx + c$$

with a, b, c real numbers. We solve for the roots of the quadratic by using the quadratic formula. If $ax^2 + bx + c = 0$, then we obtain the roots by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In the past, we only took real roots. Now we can use complex roots. For example, the roots of $x^2 + 1 = 0$ are $x = i$ and $x = -i$. We can check by multiplying the factors together:

$$(x - i)(x + i) = x^2 + xi - xi - i^2 = x^2 + 1$$

Some "Facts" about polynomials where we allow complex roots:

1. An n^{th} degree polynomial can always be factored into n roots. (Unlike if we only have real roots!) This is the *Fundamental Theorem of Algebra*.

2. If $a+bi$ is a root to a real polynomial, then $a-bi$ must also be a root. This is sometimes referred to as "roots must come in conjugate pairs". For example, if we are finding roots to a third degree polynomial, then we only have the choices that:

   (a) One root is real, the other two are complex.

   (b) All roots are real.

   And in the case of the quadratic polynomial, either:

   (a) Both roots are real.

   (b) Both roots are complex (and are conjugates of each other).

3 Exercises

1. Suppose the roots to a cubic polynomial are $a = 3$, $b = 1 - 2i$ and $c = 1 + 2i$. What is $(x - a)(x - b)(x - c)$? Will the coefficients be real?

2. Find the roots to $x^2 - 2x + 10$. Write them in polar form.

3. Let $z = a + bi$. Show that $|z|^2 = z\bar{z}$. This shows that if we multiply a complex number by it's conjugate, we get a real number!

4. For the following, let $z_1 = 3 + 2i$, $z_2 = -4i$, $z_3 = 2e^{\pi/4i}$

   (a) Compute $z_1 + z_2$, $z_1\bar{z}_2$, z_2/z_1

   (b) Write z_3 in the form $a + bi$.

   (c) Write z_1 and z_2 in polar form.

   (d) Suppose that z is written in polar form, $re^{i\theta}$. How would $\bar{z}$ be written?

5. Compute i^2, i^3, i^4, i^5, etc. Do you see a pattern?

6. In each problem, rewrite each of the following in the form $a + bi$:

   (a) e^{1+2i}

   (b) e^{2-3i}

   (c) $e^{i\pi}$

   (d) 2^{1-i}

   (e) $e^{2-\frac{\pi}{2}i}$

   (f) π^i