# Overview of Integration Techniques

Summary: We review several common integration procedures here. They are: (1) Integration by parts, (2) Partial Fractions, (3) Rationalizing substitutions. We won't do trig substitutions very often, but it might also be helpful to you to review those, also.

1. Integration by Parts, using a table.

   Standard integration by parts uses the following statement:

$$\int u\,dv = uv - \int v\,du$$

   but we will use the following table:

| sign | $u$ | $dv$ |
|---|---|---|
| | | |

   The table works by: (1) Put alternating signs in column 1, starting with a plus. (2) Differentiate the column with $u$. (3) Antidifferentiate the column with $dv$. We can then compute the integral by going straight across to get the sign, then on the diagonal. Here are some examples:

   - $$\int t^2 e^{-2t}\,dt$$

| sign | $u$ | $dv$ |
|---|---|---|
| $+$ | $t^2$ | $e^{-t}$ |
| $-$ | $2t$ | $\frac{-1}{2}e^{-t}$ |
| $+$ | $2$ | $\frac{1}{4}e^{-t}$ |
| $-$ | $0$ | $\frac{-1}{8}e^{-t}$ |

   So that:

$$\int t^2 e^{-2t}\,dt = \frac{-1}{2}t^2 e^{-t} + \frac{-1}{2}te^{-t} + \frac{-1}{4}e^{-t}$$

   - $$\int e^{-t}\sin(2t)\,dt$$

| sign | $u$ | $dv$ |
|---|---|---|
| $+$ | $e^{-t}$ | $\sin(2t)$ |
| $-$ | $-e^{-t}$ | $\frac{-1}{2}\cos(2t)$ |
| $+$ | $e^{-t}$ | $\frac{-1}{4}\sin(2t)$ |

   So that:

$$\int e^{-t}\sin(2t)\,dt = \frac{-1}{2}e^{-t}\cos(2t) + \frac{-1}{4}e^{-t}\sin(2t) - \frac{1}{4}\int e^{-t}\sin(2t)\,dt$$

2. Partial Fractions.

   We use partial fractions when a fractional integrand has a denominator that can be factored. The idea is then to break the fraction up into simpler sums. For a full discussion, see any Calculus text. We will do some of the more common examples below:

   - $$\int \frac{1}{y(1-y)}\,dy$$

     We set:

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$$

and find the appropriate constants $A, B$:

$$\frac{A}{y} + \frac{B}{1-y} = \frac{A(1-y) + By}{y(1-y)} = \frac{1}{y(1-y)}$$

which means that $A - Ay + By = 1$ for all $y$. Thus, we have two equations. One equation for the constant terms, and one equation for the coefficients of $y$:

$$\begin{aligned} A &= 1 \\ -A + B &= 0 \end{aligned}$$

So that $A = 1$, $B = 1$. Now integrate:

$$\int \frac{1}{y(1-y)} = \int \frac{1}{y}\, dy + \int \frac{1}{1-y}\, dy = \ln|y| - \ln|1-y| + C = \ln\left(\frac{y}{y-1}\right) + C$$

- $$\int \frac{x+3}{(x+1)(x-2)}\, dx$$

  Set:

  $$\frac{x+3}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

  so that $Ax + 2A + Bx + B = x + 3$, which is actually two equations:

  $$\begin{aligned} 2A + B &= 3 \\ A + B &= 1 \end{aligned}$$

  The first equation equates the constants, and the second equation equates the coefficients of $x$. Solve these for $A, B$ and go back and integrate.

NOTE: All of the above examples had to do with linear factors in the denominator. If there is a quadratic factor, the initial substitution is different. Again, you should consult your calculus book for more information.

You should also be aware of the difference between the examples above, and the following example:

$$\int \frac{x^2}{x^2 - 4}\, dx$$

By long division, we see that:

$$\frac{x^2}{x^2-4} = 1 + \frac{4}{x^2-4}$$

And now perform partial fractions on $\frac{4}{x^2-4}$.

3. Rationalizing substitutions.

   The idea here is to remove radical signs from the integrand. An example is given below:

   - $$\int \frac{\sqrt{x+4}}{x}\, dx$$

     Use a $u, du$ substitution, where $u = \sqrt{x+4}$. Then $u^2 = x + 4$, and $2u\, du = dx$. Back substituting, we get that:

     $$\int \frac{\sqrt{x+4}}{x}\, dx = \int \frac{2u^2}{u^2-4}\, du$$

     which we can integrate by the method shown previously.