Section 2.6, Problem 26 Write Up

Problem: Given that

$$y' + \frac{2}{3}y = 1 - \frac{1}{2}t, \;\; y(0) = y_0$$

Find the value of y_0 so that the solution just touches, but does not cross, the $t-$axis.

SOLUTION:

First, if $y(t)$ touches, but does not cross, the $t-$axis at some point, $t*$, then:

$$y(t*) = 0, \;\; y'(t*) = 0 \;\; y''(t*) \neq 0$$

Secondly, the differential equation, $y' + \frac{2}{3}y = 1 - \frac{1}{2}t$, must hold for y at all t, and in particular, for $t*$.

Putting this together,

$$0 + \frac{2}{3}0 = 1 - \frac{1}{2}t* \quad \Rightarrow \quad t* = 2$$

Now, our solution also satisfies: $y(2) = 0$ and $y'(2) = 0$.

We solve the differential equation using an integrating factor to get:

$$\left(y\mathrm{e}^{\frac{2t}{3}}\right)' = \mathrm{e}^{\frac{2t}{3}} - \frac{1}{2}t\mathrm{e}^{\frac{2t}{3}}$$

Integrate the right hand side by Integration by Parts to get:

$$y\mathrm{e}^{\frac{2t}{3}} = \frac{3}{2}\mathrm{e}^{\frac{2t}{3}} - \frac{1}{2}\left(\frac{3}{2}t\mathrm{e}^{\frac{2t}{3}} - \frac{9}{4}\mathrm{e}^{\frac{2t}{3}}\right) + C$$

Simplifying, we get that:

$$y = \frac{21}{8} - \frac{3}{4}t + C\mathrm{e}^{-\frac{2}{3}t}$$

and, using $y(0) = y_0$, we get that $C = y_0 - \frac{21}{8}$. So,

$$y = \frac{21}{8} - \frac{3}{4}t + (y_0 - \frac{21}{8})\mathrm{e}^{-\frac{2}{3}t}$$

Finally, using $y(2) = 0$, we solve for y_0:

$$0 = \frac{21}{8} - \frac{3}{4}2 + (y_0 - \frac{21}{8})\mathrm{e}^{-\frac{2}{3}2}$$

From which we get that:

$$y_0 = \frac{3}{2}\mathrm{e}^{4/3} + \frac{21}{8}(1 - \mathrm{e}^{4/3}) \approx -1.64$$