Section 7.1

Goals for 7.1:

- Be able to convert an n^{th} order differential equation to a system of first order differential equations, and (where possible) convert a system of two differential equations to a single second order differential equation.
- Understand where systems of differential equations come from. We've seen: Predator-Prey, system of springs, system of tanks. In general, any time we're modeling the interaction of states, we'll get a system of differential equations.
- Understand the extension of *equilibria* to systems of autonomous differential equations.

Problems 7.1, #2, 5, 14, 21

2. Convert to a system of first degree equations: $u'' + 0.5u' + 2u = 2\sin(t)$. SOLUTION: Let $x = u, y = u'$ Then:

$$\begin{aligned} x' &= y \\ y' &= -2x - 0.5y + 3\sin(t) \end{aligned}$$

5. Convert to a system of first order differential equations with the initial value: $u'' + p(t)u' + q(t)u = g(t)$, $u(0) = u_0$, $u'(0) = u'_0$. SOLUTION: Let $x = u$, $y = u'$. Then:

$$\begin{aligned} x' &= y \\ y' &= -q(t)x - p(t)y + g(t) \end{aligned}$$

with initial values: $x(0) = u_0$, $y(0) = u'_0$

14. Put the following system into a single second order equation.

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + g_1(t) \\ x_2' &= a_{21}x_1 + a_{22}x_2 + g_2(t) \end{aligned}$$

We solve the first equation for x_2, use it to get an expression for x_2', then substitute these expressions into the second equation:

If $a_{12} \neq 0$, then we get the following for x_2 and x_2':

$$\begin{aligned} x_2 &= \tfrac{1}{a_{12}}\left(x_1' - a_{11}x_1 - g_1(t)\right) \\ x_2' &= \tfrac{1}{a_{12}}\left(x_1'' - a_{11}x_1' - g_1'(t)\right) \end{aligned}$$

Substituting these expressions into the second original equation yields:

$$\frac{1}{a_{12}}\left(x_1'' - a_{11}x_1' - g_1'(t)\right) = a_{21}x_1 + a_{22}\frac{1}{a_{12}}\left(x_1' - a_{11}x_1 - g_1(t)\right) + g_2(t)$$

Simplifying, we get:

$$x_1'' - (a_{11} + a_{22})x_1' + (a_{11}a_{22} - a_{21}a_{12})x_1 = g_1'(t) - a_{22}g_1(t) + a_{12}g_2(t)$$

If $a_{12} = 0$, we would perform a similar procedure, but use the second equation and solve for x_1. Here we would need to assume that $a_{21} \neq 0$

The same procedure can be carried out if a_{ij} are functions of time, but we would want to be sure that division would not be by zero.

NOTE: We cannot start this problem by assuming that $x_1 = u$ and $x_2 = u'$. This would imply that $x_1' = x_2$, which is probably not true, given that

$$x_1' = a_{11}x_1 + a_{12}x_2 + g_1(t)$$

21. We started this one in class. From the model of the two tanks, we get the following system of differential equations:

$$\begin{aligned} Q_1' &= 1.5 + \tfrac{1.5}{20}Q_2 - \tfrac{3}{30}Q_1 \\ Q_2' &= 3 + \tfrac{3}{30}Q_1 - \tfrac{4}{20}Q_2 \end{aligned}$$

Hint if you're not sure how we got this: Be sure your units of measure are lining up. Note that, since Q_1 and Q_2 are in ounces, and the time is given is minutes, you should have that Q_1' and Q_2' are being measured in ounces per minute.

We find equilibrium where $Q_1' = 0$ and $Q_2' = 0$: In matrix form, we solve:

$$\begin{bmatrix} \frac{-1}{10} & \frac{3}{40} \\ \frac{1}{10} & \frac{-1}{5} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -3 \end{bmatrix}$$

so we get the equilibria at $Q_1^E = 42, Q_2^E = 36$. Lastly, let $x_1 = Q_1 - Q_1^E$, $x_2 = Q_2 - Q_2^E$. Then:

$$x_1' = Q_1' = 1.5 + \frac{1.5}{20}Q_2 - \frac{3}{30}Q_1 = 1.5 + \frac{1.5}{20}(x_2 + 36) - \frac{3}{30}(x_1 + 42) = \frac{-1}{10}x_1 + \frac{3}{40}x_2$$

and

$$x_2' = Q_2' = 3 + \frac{3}{30}Q_1 - \frac{4}{20}Q_2 = 3 + \frac{3}{30}(x_1 + 42) - \frac{4}{20}(x_2 + 36) = \frac{1}{10}x_1 + \frac{-1}{5}x_2$$

Section 7.2

Goals for 7.2:

- Recall the following matrix and vector operations (especially 2×2 and 3×3): $A^T, AB, A\boldsymbol{x}$
- Be able to solve $A\boldsymbol{x} = \boldsymbol{b}$, especially if A is 2×2 or 3×3, and especially if $\boldsymbol{b} = \boldsymbol{0}$.
- New operations: $\boldsymbol{x}'(t)$, $A'(t)$, $\displaystyle\int_a^b A(t)\, dt$
- Product Rule: $(AB)'(t) = A'(t)B(t) + A(t)B'(t)$, and $(A\boldsymbol{x})'(t) = A'(t)\boldsymbol{x}(t) + A(t)\boldsymbol{x}'(t)$
- Be able to verify that something is a solution to a given differential equation.

Problems 7.2, #22, 23, 26

22.

$$\begin{pmatrix} 7e^t & 5e^{-t} & 10e^{2t} \\ -e^t & 7e^{-t} & 2e^{2t} \\ 8e^t & 0 & -e^{2t} \end{pmatrix}, \quad \begin{pmatrix} 2e^{2t} - 2 + 3e^{3t} & 1 + 4e^{-2t} - e^t & 3e^{3t} + 2e^t - e^{-4t} \\ 4e^{2t} - 1 - 3e^{3t} & 2 + 2e^{-2t} + e^t & 6e^{3t} + e^t + e^{-4t} \\ -2e^{2t} - 3 + 6e^{3t} & -1 + 6e^{-2t} - 2e^t & -3e^{3t} + 3e^t - 2e^{-4t} \end{pmatrix}$$

$$\begin{pmatrix} e^t & -2e^{-t} & 2e^{2t} \\ 2e^t & -e^{-t} & -2e^{2t} \\ -e^t & -3e^{-t} & 4e^{2t} \end{pmatrix}, \quad \begin{pmatrix} (e-1) & 2(1-e^{-1}) & \frac{1}{2}(e^2-1) \\ 2(e-1) & (1-e^{-1}) & -\frac{1}{2}(e^2-1) \\ -(e-1) & 3(1-e^{-1}) & (e^2-1) \end{pmatrix}$$

23. First, compute $\boldsymbol{x}'$ using the given $\boldsymbol{x}$, then compare with $A\boldsymbol{x}$:

$$\boldsymbol{x}' = \begin{bmatrix} 4 \\ 2 \end{bmatrix} 2e^{2t} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} e^{2t}$$

and:

$$A\boldsymbol{x} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{2t} = \begin{bmatrix} 12 - 4 \\ 8 - 4 \end{bmatrix} e^{2t} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} e^{2t}$$

26. First, compute $\boldsymbol{\Psi}'$ using the given $\boldsymbol{\Psi}$, then compare with $A\boldsymbol{\Psi}$:

$$\boldsymbol{\Psi}' = \begin{bmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{bmatrix}$$

and:

$$A\boldsymbol{\Psi} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{bmatrix} = \begin{bmatrix} (1-4)e^{-3t} & (1+1)e^{2t} \\ (4+8)e^{-3t} & (4-2)e^{2t} \end{bmatrix} = \begin{bmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{bmatrix}$$

Section 7.3

Goals for 7.3:

- Understand *linear independence*, and the difference between linearly independent *vectors* and linearly independent *solutions*.
- Recall how to compute eigenvalues and eigenvectors:
  1. Solve for λ: $\det(A - \lambda I) = 0$
  2. For each λ, solve for $\boldsymbol{v}$: $(A - \lambda I)\boldsymbol{v} = 0$.
- Understand what it means to diagonalize a matrix.

Problems 7.3, #14, 25

14. Let $\boldsymbol{x}^{(1)} = (\mathrm{e}^t, t\mathrm{e}^t)^T, \boldsymbol{x}^{(2)} = (1, t)^T$ Show that $\boldsymbol{x}^{(1)}(t), \boldsymbol{x}^{(2)}(t)$ are linearly dependent for any fixed t:

$$W[\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}](t) = \begin{vmatrix} \mathrm{e}^t & 1 \\ t\mathrm{e}^t & t \end{vmatrix} = \mathrm{e}^t(t - t) = 0$$

NOTE: The Wronskian is checking for linear dependence *pointwise*. If the Wronskian for two functions is non-zero somewhere, then that suffices to say that the functions are linearly independent. If the Wronskian is zero for all t, we cannot say anything about independence.

For the full interval, we are trying to solve:

$$c_1 \begin{bmatrix} \mathrm{e}^t \\ t\mathrm{e}^t \end{bmatrix} c_2 \begin{bmatrix} 1 \\ t \end{bmatrix} = \mathbf{0}, \text{ for all t}$$

But this implies that $c_1\mathrm{e}^t + c_2 = 0$ for all t, so $c_1 = 0$. If $c_1 = 0$, $c_2 = 0$. Thus, the only solution is trivial.

This is a nice example showing the difference between linearly independent vectors and linearly independent solutions.

25. For each problem, T is the matrix of eigenvectors, and D is the matrix with eigenvalues along the diagonal. Below are listed the eigenvalues and eigenvectors.

Although there are a lot of computations involved in these problems, they combine everything we need to know about eigenvalues, eigenvectors, and diagonalization. It's therefore a very useful exercise to check your computational abilities.

(a) (15):

$$T = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix},$$

(b) (16):

$$T = \begin{bmatrix} 1 & 1 \\ 1 - i & 1 + i \end{bmatrix}, \quad D = \begin{bmatrix} 1 + 2i & 0 \\ 0 & 1 - 2i \end{bmatrix},$$

(c) (18):

$$T = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix},$$