Homework: 6.1, 6.2, 6.3

6.1, #19 Compute $\mathcal{L}(t^2\sin(at))$. Note the following, from Euler's Formula:

$$t^2 \mathrm{e}^{iat} = t^2\cos(at) + it^2\sin(at)$$

so that, by the linearity of the Laplace transform:

$$\mathcal{L}(t^2\mathrm{e}^{iat}) = \mathcal{L}(t^2\cos(at)) + i\mathcal{L}(t^2\sin(at))$$

so that we need to compute $\mathcal{L}(t^2\mathrm{e}^{iat})$, then find the imaginary part of our answer.

Now:

$$\mathcal{L}(t^2\mathrm{e}^{iat}) = \int_0^\infty t^2\mathrm{e}^{-t(s-ia)}\,dt$$

so that integration by parts yields:

$$\left.\frac{-t^2}{s-ia}\mathrm{e}^{-t(s-ia)}\right|_0^\infty - \left.\frac{2t}{(s-ia)^2}\mathrm{e}^{-t(s-ia)}\right|_0^\infty - \left.\frac{2}{(s-ia)^3}\mathrm{e}^{-t(s-ia)}\right|_0^\infty$$

After evaluating the limits, we see that:

$$\mathcal{L}(t^2\mathrm{e}^{iat}) = \frac{2}{(s-ia)^3}$$

which is, after multiplying top and bottom by $(s+ia)^3$,

$$\mathcal{L}(t^2\mathrm{e}^{iat}) = \frac{2s^3-6a^2s}{(s^2+a^2)^3} + i\frac{6as^2-2a^3}{(s^2+a^2)^3}$$

Therefore,

$$\mathcal{L}(t^2\sin(at)) = \frac{6as^2-2a^3}{(s^2+a^2)^3}$$

NOTE: We can verify this by computing the Laplace transform by using Table Entry 19: $(-t)^n f(t) \leftrightarrow F^{(n)}(s)$

6.1, #21 A straightforward computation shows this integral converges:

$$\int_0^\infty \frac{1}{1+t^2}\,dt = \lim_{T\to\infty} \tan^{-1}(t)\Big|_0^T = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

6.1, #23 Does the integral converge or diverge? $\displaystyle\int_1^\infty \frac{\mathrm{e}^t}{t^2}\,dt$

First, we note that:

$$\lim_{t\to\infty}\frac{\mathrm{e}^t}{t^2} = \lim_{t\to\infty}\frac{\mathrm{e}^t}{2t} = \lim_{t\to\infty}\frac{\mathrm{e}^t}{2} = \infty$$

Therefore, we suspect that the integral diverges. However, we will not be able to find a simple antiderivative. We must instead compare this integral to an integral that is known to diverge. (Side Remark: Note that $\displaystyle\int_1^\infty \frac{1}{t^2}\,dt = 1$, so this function won't work).

From our previous computation, we know that $\frac{e^t}{t^2}$ goes to infinity. If we can show that, for some constant M and some time T, $\frac{e^t}{t^2} > M$ for all $t \geq T$, then we will be almost done!

If $f(t) = \frac{e^t}{t^2}$, then $f'(t) = \frac{t-2}{t^3}e^t$. THIS IS VERY NICE! It says that $f(t)$ is increasing for all time after $t = 2$. Therefore, $f(t) \geq f(2)$, for all $t \geq 2$. Therefore:

$$\frac{e^t}{t^2} > \frac{e^2}{4}, \text{ for all } t > 2$$

so that:

$$\int_2^\infty \frac{e^t}{t^2}\, dt > \int_2^\infty \frac{e^2}{4}\, dt = \frac{e^2}{4}\int_2^\infty 1\, dt$$

The last integral above diverges. Our conclusion:

The integral $\displaystyle\int_1^\infty \frac{e^t}{t^2}\, dt$ diverges.

6.2, #8 Compute the inverse Laplace transform.

$$\frac{8s^2 - 4s + 12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs + C}{s^2+4}$$

where:

$$\begin{aligned} A + B &= 0 \\ C &= -4 \\ 4A &= 12 \end{aligned}$$

From which $A = 3, B = 5, C = -4$, so:

$$\frac{8s^2 - 4s + 12}{s(s^2+4)} = 3\frac{1}{s} + 5\frac{s}{s^2+4} - 2\frac{2}{s^2+4}$$

so that the inverse transform is: $3 + 5\cos(2t) - 2\sin(2t)$

6.2, #15 Solve $y'' - 2y' - 2y = 0, y(0) = 2, y'(0) = 0$. First, we compute the transform of both sides to get:

$$s^2Y - 2s - 2(sY - 2) - 2Y = 0$$

$$Y = \frac{2s-4}{s^2-2s-2}$$

We note that the roots of $s^2 - 2s - 2$ are $-1 \pm \sqrt{3}$, which is messy to work with. We will instead complete the square and use the table entries with hyperbolic sines and cosines (Note: Only do this as a "last resort", always try to get nice factors first).

$$\frac{2s-4}{s^2-2s-2} = 2\frac{(s-1)}{(s-1)^2-3} - \frac{2}{\sqrt{3}}\frac{\sqrt{3}}{(s-1)^2-3}$$

so that

$$y(t) = e^t\left(2\cosh(\sqrt{3}t) - \frac{2}{\sqrt{3}}\sinh(\sqrt{3}t)\right)$$

6.3, #6 Sketch $f(t) = u_1(t)(t-1) - u_2(t)2(t-2) + u_3(t)(t-3)$

To begin, write $f(t)$ piecewise:

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & 1 \le t < 2 \\ (t-1) - 2t + 2, & 2 \le t < 3 \\ (t-1) - 2t + 2 + t - 3, & t \ge 3 \end{cases}$$

which simplifies to:

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & 1 \le t < 2 \\ 3-t, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

6.3, #14 Invert: $\frac{e^{-2s}}{s^2+s-2}$

First, think of this as $e^{-2s}H(s)$, so that the inverse transform is:

$$u_2(t)h(t-2)$$

Now we need to find $h(t)$:

See if the denominator is easily factorable, and in this case it is: $s^2+s-2 = (s+2)(s-1)$. Therefore,

$$H(s) = \frac{1}{s^2+s-2} = \frac{A}{s+2} + \frac{B}{s-1}$$

When we solve for A, B, we get: $A = -\frac{1}{3}, B = \frac{1}{3}$. Therefore,

$$h(t) = \frac{-1}{3}e^{-2t} + \frac{1}{3}e^{t}$$

6.3, #20 This problem could be solved by a variety of means- the answer is $2(2t)^n$