Laplace Transforms Summary

1 Introductory and Overview

Integral Transforms are used to analyze functions. For example, Fourier transforms are used to compare functions to sines and cosines, while Laplace Transforms can be used to compare functions to the exponential function (note what happens to the Laplace transform of e^3t as $s \to 3$).

The main feature in using Laplace Transforms for differential equations is the fact that transforming the derivative results in an expression with no derivatives. Therefore, the Laplace transform used on linear differential equations with constant coefficients changes a differential equation into an algebraic equation, which can, in principle, be easily solved.

Summary of the details:

- $\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$. Be able to use this definition to compute Laplace transforms without the table. Note this includes understanding what the improper integral represents.
- Existence of the Laplace Transform: $f(t)$ should be piecewise continuous (no vertical asymptotes), and should be of exponential order.
- f is of exponential order if we can find constants M, k, T such that $|f(t)| \le Me^{kt}, t \ge T$. In words, f cannot grow faster than an exponential function.

The rest of Chapter 6 is devoted to understanding the Table in Section 6.2.

2 New Functions/Operations

1. The Heaviside Function: $u_c(t)$.
   - Know the definition of $u_c(t)$.
   - $u_a(t) - u_b(t)$ turns ON at $t = a$ and turns OFF at $t = b$.
   - Be able to translate a function using a Heaviside function to a piecewise defined function. Be able to translate a piecewise defined function into a function using the Heaviside function.
   - Be able to go from the graph of a piecewise defined function to its expression involving the Heaviside function, and vice versa.
2. The Dirac $\delta-$function. Understand the definition of the $\delta-$function and its use in differential equations.

   The Properties of the $\delta-$function to remember:
   - $\int_0^\infty \delta(t-c)\, dt = 1$
   - $\int_0^\infty \delta(t-c) f(t)\, dt = f(c)$
   - The "derivative" of $u_c(t)$ is $\delta(t-c)$.
3. New Operation: Convolution.

   Definition: The convolution of $f(t)$ and $g(t)$ is:

$$(f * g)(t) = \int_0^t f(\lambda) g(t-\lambda)\, d\lambda$$

The main way we use the convolution is to invert products of Laplace transforms:
If $F(s) = G(s)H(s)$, the $f(t) = (g * h)(t)$.

NOTE: Only use this when you *have* to, and not to avoid partial fractions (unless specifically stated otherwise).

We can also use Laplace transforms to help us evaluate the convolution operation:

$$f * g \Rightarrow F(s)G(s) \Rightarrow \text{Expression in s} \Rightarrow \text{Expression in t}$$

3 Methods to Remember and some Tricks

1. Algebra:

   - Complete the square: $as^2 + bs + c = a(s^2 + \frac{b}{a}s + \quad\quad) + c$ Fill in the blanks with $(\frac{b}{2a})^2$, so that:

     $$as^2 + bs + c = a(s + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

     (A neat exercise: Derive the quadratic formula from the formula for completing the square above).
   - Factoring over the reals: If $s = a$ is a root of a polynomial in s, then $(s - a)$ can be factored out (The Fundamental Theorem of Algebra).
   - Partial Fractions (See the handout on Partial Fractions).
   - Function notation: $f(t), f(t - c), F(s), F(s - c)$, etc.

2. Some hints:

   - If $as^2 + bs + c$ cannot be factored over the reals, we must complete the square. This will result in needing to use the shift formulas, $F(s - c)$.
   - If $as^2 + bs + c$ is not easily factored over the reals, (i.e., $s = -\frac{1}{2} \pm \sqrt{5}$), complete the square and use the hyperbolic sine and cosine entries of the table.
   - Bookkeeping: If you're dealing with:

     $$\mathrm{e}^{-cs} \cdot \text{Expression in s}$$

     make the expression in s: $H(s)$, then find $h(t)$ seperately. At the end, make the solution: $y(t) = u_c(t)h(t - c)$, without actually re-writing $h(t)$ over again. This method works for more complicated expressions, such as $(\mathrm{e}^{-ks} - \mathrm{e}^{-js})H(s)$, as well.
   - When inverting transforms, only use the convolution entry of the table as a last resort.
   - Add and subtract the same number:

     $$\frac{s}{(s-3)^2 + 4} = \frac{s-3}{(s-3)^2 + 4} + \frac{3}{(s-3)^2 + 4}$$

   - Multiply and divide the same number:

     $$\frac{3}{(s-3)^2 + 4} = \frac{3}{2}\left(\frac{2}{(s-3)^2 + 4}\right)$$

   - Dealing with $u_c(t)f(t - c)$ when you've got $u_c(t)g(t)$: Let $f(t - c) = g(t)$, so that $f(t) = g(t + c)$. Example: Transform $tu_3(t)$. Let $f(t - 3) = t$, so $f(t) = t + 3$. Then, the table says to get $F(s)$, which in this case is $\frac{1}{s^2} + \frac{3}{s}$, so the full transform is $\mathrm{e}^{-3s}(\frac{1}{s^2} + \frac{3}{s})$.
     Hopefully, this will still be easy to transform. Also, see the Miscellaneous section on the practice problems.