Review Questions: Laplace Transforms

1. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:

   (a)
   $$f(t) = \begin{cases} 3, & 0 \le t \le 2 \\ 6-t, & 2 < t \end{cases}$$

   (b)
   $$f(t) = \begin{cases} e^{-t}, & 0 \le t \le 5 \\ -1, & t > 5 \end{cases}$$

2. Determine the Laplace transform:

   (a) $t^2 e^{-9t}$
   (b) $e^{2t} - t^3 - \sin(5t)$
   (c) $u_5(t)(t-5)^4$
   (d) $e^{3t}\sin(4t)$
   (e) $e^t \delta(t-3)$
   (f) $t^2 u_4(t)$
   (g) $\displaystyle\int_0^t \sin(t-\lambda) e^{\lambda}\, d\lambda$

3. Find the inverse Laplace transform:

   (a) $\dfrac{2s-1}{s^2-4s+6}$
   (b) $\dfrac{s^2+16s+9}{(s+1)(s+3)(s-2)}$
   (c) $\dfrac{7}{(s+3)^3}$
   (d) $\dfrac{e^{-2s}(4s+2)}{(s-1)(s+2)}$
   (e) (Use convolutions): $\frac{1}{s^3(s^2+1)}$
   (f) $\dfrac{3s-2}{(s-4)^2-3}$

4. Solve the given initial value problems using Laplace transforms:

   (a) $y'' - 7y' + 10y = 0$, $y(0) = 0$, $y'(0) = -3$
   (b) $y'' + 6y' + 9y = 0$, $y(0) = -3$, $y'(0) = 10$
   (c) $y'' + 2y' + 2y = t^2 + 4t$, $y(0) = 0$, $y'(0) = -1$
   (d) $y'' + 9y = 10e^{2t}$, $y(0) = -1$, $y'(0) = 5$
   (e) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$
   (f) $y'' - 4y' + 4y = t^2 e^t$, $y(0) = 0$, $y'(0) = 0$
   (g) $y' - 2\displaystyle\int_0^t y(\lambda)\sin(t-\lambda)\, d\lambda = 1, y(0) = -1$
   (h) $y'' + 4y = \delta(t - \frac{\pi}{2})$, $y(0) = 0$, $y'(0) = 1$
   (i) $y'' + y = \displaystyle\sum_{k=1}^{\infty} \delta(t - 2k\pi)$, with initial conditions both 0.

5. Miscellaneous Problems:

   (a) Evaluate: $\displaystyle\int_0^{\infty} \sin(3t)\delta(t - \frac{\pi}{2})\, dt$
   (b) Evaluate, using Laplace transforms: $\sin(t) * t$
   (c) If f is periodic with period T, show that
   $$\mathcal{L}(f) = \frac{\displaystyle\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}$$
   (d) Use the table to find an expression for $\mathcal{L}(ty')$. Use this to solve:
   $$y'' + 3ty' - 6y = 1, \quad y(0) = 0, \quad y'(0) = 0$$

6. Characterize ALL solutions to $y'' + 4y = u_1(t-1)$, $y(0) = 1$, $y'(0) = 1$.

7. Define $\delta(t-c)$ as a limit of "regular" functions.

8. If $y'(t) = \delta(t-c)$, what is $y(t)$?

9. Show that $\mathcal{L}(g(t+c)) = e^{cs}(G(s) - \int_0^c e^{-st} g(t)\, dt)$. This might be useful, as mentioned in the notes on inverting $f(t)u_c(t)$.