Laplace Transforms Review Solutions

1. Compute transforms from the definition:

   (a) $\int_0^2 3e^{-st}\,dt + \int_2^\infty (6-t)e^{-st}\,dt = \dfrac{3}{s} + \dfrac{5}{s}e^{-2s} + \dfrac{1}{s^2}e^{-2s}$

   (b) $\int_0^5 e^{-t}e^{-st}\,dt - \int_5^\infty e^{-st}\,dt = \dfrac{1-e^{-5(s+1)}}{s+1} - \dfrac{e^{-5s}}{s}$

2. Compute transforms (using the table)

   (a) $\frac{2}{(s+9)^3}$

   (b) $\frac{1}{s-2} - \frac{6}{s^4} - \frac{5}{s^2+25}$

   (c) $e^{-5s}\frac{4!}{s^5}$

   (d) $\frac{4}{(s-3)^2+16}$

   (e) Use Table #14, with $f(t) = \delta(t-3)$, so $e^{-3(s-1)}$.

   (f) Let $f(t-4) = t^2$, so $f(t) = (t+4)^2$: $e^{-4s}\left(\frac{2}{s^3} + \frac{8}{2^2} + \frac{16}{s}\right)$

   (g) $F(s)G(s) = \frac{1}{(s-1)(s^2+1)}$

3. Invert the transforms:

   (a) First rewrite as $\frac{2s-1}{(s-2)^2+2}$, so $2e^{\sqrt{2}t}\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}}e^{2t}\sin(\sqrt{2}t)$

   (b) Via partial fractions: $e^{-t} - 3e^{-3t} + 3e^{2t}$

   (c) $\frac{7}{2}e^{-3t}t^2$

   (d) $u_2(t)[2e^{-2(t-2)} + 2e^{t-2}]$

   (e) $\int_0^t \frac{1}{2}(t-\lambda)^2\sin(\lambda)\,d\lambda$, or $\frac{1}{2}t^2 * \sin(t)$

   (f) Rewrite to get: $3\frac{(s-4)}{(s-4)^2+3} + \frac{10}{\sqrt{3}}\frac{\sqrt{3}}{(s-4)^2+3}$, so we get: $3e^{4t}\cos(\sqrt{3}t) + \frac{10}{\sqrt{3}}e^{4t}\sin(\sqrt{3}t)$

4. Solve the diff. eqn:

   (a) $e^{2t} - e^{5t}$

   (b) $-3e^{-3t} + te^{-3t}$

   (c) $\frac{1}{2}t^2 + t - \frac{3}{2} + e^{-t}(\frac{3}{2}\cos(t) - \frac{1}{2}\sin(t))$

   (d) $\frac{10}{13}e^{2t} - \frac{23}{13}\cos(3t) + \frac{15}{13}\sin(3t)$

   (e) Let $h(t) = \frac{-1}{3} + \frac{1}{12}e^{3t} + \frac{1}{4}e^{-t}$ Then the solution is: $y(t) = u_1(t)h(t-1) - \frac{1}{4}e^{3t} + \frac{1}{4}e^{-t}$.

   (f) $e^{2t}(2t-6) + e^{t}(t^2+4t+6)$

   (g) The Laplace Transform should yield:

   $$Y(s) = \frac{s^2+1}{s(s^3+s-2)} - \frac{1}{s^3+s-2}$$

   and $s-1$ divides s^3+s-2. $y(t) = -\frac{1}{2} + e^{-\frac{1}{2}t}\left(\frac{3}{\sqrt{7}}14\sin(\frac{\sqrt{7}}{2}t) - \frac{1}{2}\cos(\frac{\sqrt{7}}{2}t)\right)$

(h) $\frac{1}{2}\sin(2t) + \frac{1}{2}u_{\frac{\pi}{2}}(t)\sin(2(t-\frac{\pi}{2}))$

(i) $y(t) = \sum_{k=1}^{\infty} u_{2\pi k}(t)\sin(t)$ Note that this is:

$$y(t) = \begin{cases} \sin(t), & 0 \le t < 2\pi \\ 2\sin(t), & 2\pi \le t < 4\pi \\ 3\sin(t), & 4\pi \le t < 6\pi \\ \vdots & \vdots \end{cases}$$

5. Miscellaneous:

(a) Recall that $\int_0^\infty \delta(t-c)f(t)\,dt = f(c)$, so this is $\sin(3\pi/2) = -1$.

(b) $L(\sin(t) * t) = \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$ so $\sin(t) * t = t - \sin(t)$

(c)

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \dots$$

Let $u_1 = t - T$, $u_2 = t - 2T$, $u_3 = t - 3T$, etc. to get:

$$\int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u_1+T)} f(u_1)\,du_1 + \int_0^T e^{-s(u_2+2T)} f(u_2)\,du_2 + \int_0^T e^{-s(u_3+3T)} f(u_3)\,du_3 + \dots$$

which, when factored, yields:

$$\int_0^T e^{-st} f(t)\,dt \left[1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \dots\right]$$

The sum in the brackets is a geometric series, so when we sum it we get:

$$\int_0^T e^{-st} f(t)\,dt \frac{1}{1 - e^{-sT}}$$

For example, if $f(t) = \sin(t)$, then $T = 2\pi$, and

$$\int_0^{2\pi} e^{st} \sin(t)\,dt = \frac{1 - e^{-2\pi s}}{s^2 + 1}$$

so we get the normal table entry.

(d) $L(ty') = -L((-t)y'(t)) = -F'(s)$, where $F(s) = L(y'(t))$. so we get:

$$\begin{aligned} L(ty') &= -\tfrac{d}{ds}L(y') \\ &= -\tfrac{d}{ds}(sY - y(0)) \\ &= -sY'(s) - Y(s) \end{aligned}$$

so taking the Laplace transform of the differential equation will yield:

$$s^2Y + 3(-sY' - Y) - 6Y = \frac{1}{s}$$

or

$$Y' + \frac{s^2 - 9}{-3s}Y = \frac{-1}{3}1s^2$$

which is a linear differential equation, with an integrating factor: $s^3 e^{\frac{-1}{6}s^2}$ so we get:

$$\left(s^3 e^{\frac{-1}{6}s^2} Y\right)' = \frac{-1}{3} s e^{\frac{-1}{6}s^2}$$

from which $Y = \frac{1}{s^3}$, so $y(t) = \frac{1}{2}t^2$.

6. Characterize all solutions: We solve by our old method of getting the homogeneous and particular solutions (see the handout on discontinuous forcing functions for more details).

$$y(t) = \begin{cases} \cos(2t) + \frac{1}{2}\sin(2t), & 0 \le t < 1 \\ c_1\cos(2t) + c_2\sin(2t) + \frac{1}{4}, t \ge 1 \end{cases}$$

7. Define the delta function:

$$\delta(t-c) = \lim_{h\to 0} d_h(t-c)$$

where

$$d_h(t-c) = \begin{cases} \frac{1}{2h}, & c-h < t < c+h \\ 0, & \text{otherwise} \end{cases}$$

8. If $y'(t) = \delta(t-c)$, what is $y(t)$?

$$sY = e^{-cs} \to Y = \frac{e^{-cs}}{s} \to y(t) = u_c(t)$$

9. $\int_0^\infty e^{-st} g(t+c)\,dt = \int_c^\infty e^{-s(u-c)} g(u)\,du = e^{cs}\int_c^\infty e^{-su} g(u)\,du = e^{cs}\left(G(s) - \int_0^c e^{-st} g(t)\,dt\right)$