Summary of Ch. 5

Recall we also had a summary sheet of Chapter 9

Power series for $P(x)y'' + Q(x)y' + R(x)y = 0$.

1. Recall what a power series is, and the ratio test for convergence. Be able to change the index of the power series.
2. A point x_0 is an ordinary point if $P(x_0) \neq 0$. More generally, x_0 is an ordinary point if Q/P and R/P are analytic at x_0.
3. Be able to expand a power series solution about an ordinary point, and get a recursion relation between the coefficients. Be able to write out the first few terms of the solution.
4. Questions on convergence and the Euler Equation won't be on the exam.

General Review Questions

NOTE: You should also review past homework and exams.
This sheet also includes problems from the previous two review sheets (Ch. 1-3 and Laplace)

1. Solve (use any method if not otherwise specified):
   (a) $(2x - 3x^2)\frac{dx}{dt} = t\cos(t)$
   (b) $y'' + 2y' + y = \sin(3x)$
   (c) $(x^2 + xy)y' = x^2 + y^2$
   (d) $y'' - 3y' + 2y = \mathrm{e}^{2t}$
   (e) $xy' = y + x\cos^2\left(\frac{y}{x}\right)$
   (f) $x' = \sqrt{t}\mathrm{e}^{-t} - x$
   (g) $y'' - xy' - 2y = 0$
   (h) $x' = 2 + 2t^2 + x + t^2 x$
2. Obtain the general solution in terms of α, then determine a value of α so that $y(t) \to 0$ as $t \to \infty$:
   $$y'' - y' - 6y = 0, \quad y(0) = 1, y'(0) = \alpha$$
3. The Wronskian of two functions is $W(t) = t^2 - 4$. Are the solutions linearly independent? Why or why not?
4. Compute $\mathcal{L}(t\cos(t))$ by using the definition of the Laplace transform.
5. Write 2^i and $\frac{1-3i}{2+i}$ in $a + bi$ form.
6. Let $\mathbf{x}' = A\mathbf{x}$, where A is given below. Give a complete analysis of each, including (1) Stability classification (Poincaré), (2) Analytic solution, (3) Fundamental Matrix, (4) The Matrix Exponential (leave in factored form), and (5) Sketch the direction field.
   (a) $\begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$
   (b) $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$
   (c) $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$
   (d) $\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$

(e) $\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$

7. Let $y''' - y' = te^{-t} + 2\cos(t)$. Determine a suitable form for the particular solution, y_p. Do not solve for the coeffs.

8. Write the differential equation associated with *Resonance* and *Beating*. Discuss under what conditions we can expect each type of behavior.

9. Problem 6, p. 369.

10. Suppose that we have a mass-spring system modelled by the differential equation

$$x'' + 2x' + x = 0, x(0) = 2, x'(0) = -3$$

Find the solution, and determine whether the mass ever crosses $x = 0$. If it does, determine the velocity at that instant. See if it crosses if the velocity is cut in half.

11. How is it possible to construct a fundamental set of solutions to $\mathbf{x}' = A(t)\mathbf{x}$ if we only have a computer program that solves an initial value problem?

12. For problems 5-14, p. 478, determine the equilibria and classify stability based on the Poincaré diagram.

13. Let $y(x)$ be a power series solution to $(1-x)y'' + y = 0,\ x_0 = 0$. Find the recurrence relation, and write out the first 6 terms of y.

14. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0,\ x_0 = 1$. Find the recurrence relation and write out the first 6 terms of y.

15. True or False: If

$$dx/dt = F(x, y), \quad dy/dt = G(x, y)$$

and (x^*, y^*) is a critical point, then any other solution cannot reach the critical point in finite time. (Be sure to explain why).

16. Let $x' = \sin(y)$, $y' = \sin(x)$ Find all equilibria, and classify the stability. Sketch a picture!

17. p. 502, 1-6. Try a couple of these examples of Competing Species.

18. p. 503, 12(a-d).

19. Analyze how the origin changes classification with respect to α if:

$$\mathbf{x}' = \begin{pmatrix} 1 & \alpha \\ -\alpha & -2 \end{pmatrix} \mathbf{x}$$

20. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:

$$f(t) = \begin{cases} 3, & 0 \le t \le 2 \\ 6 - t, & 2 < t \end{cases}$$

21. Determine the Laplace transform:
    (a) $t^2 e^{-9t}$
    (b) $e^{2t} - t^3 - \sin(5t)$
    (c) $u_5(t)(t-5)^4$
    (d) $e^{3t}\sin(4t)$
    (e) $e^t \delta(t-3)$
    (f) $t^2 u_4(t)$
    (g) $\displaystyle\int_0^t \sin(t-\lambda)e^{\lambda}\, d\lambda$

22. Find the inverse Laplace transform:

(a) $\dfrac{2s-1}{s^2-4s+6}$

(b) $\dfrac{s^2+16s+9}{(s+1)(s+3)(s-2)}$

(c) $\dfrac{7}{(s+3)^3}$

(d) $\dfrac{e^{-2s}(4s+2)}{(s-1)(s+2)}$

(e) (Use convolutions): $\frac{1}{s^3(s^2+1)}$

(f) $\dfrac{3s-2}{(s-4)^2-3}$

23. Solve the given initial value problems using Laplace transforms:

(a) $y'' - 7y' + 10y = 0$, $y(0) = 0$, $y'(0) = -3$

(b) $y'' + 6y' + 9y = 0$, $y(0) = -3$, $y'(0) = 10$

(c) $y'' + 2y' + 2y = t^2 + 4t$, $y(0) = 0$, $y'(0) = -1$

(d) $y'' + 9y = 10e^{2t}$, $y(0) = -1$, $y'(0) = 5$

(e) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$

(f) $y'' - 4y' + 4y = t^2e^t$, $y(0) = 0$, $y'(0) = 0$

(g) $y' - 2\displaystyle\int_0^t y(\lambda)\sin(t-\lambda)\,d\lambda = 1, y(0) = -1$

(h) $y'' + 4y = \delta(t - \frac{\pi}{2})$, $y(0) = 0$, $y'(0) = 1$

24. Evaluate: $\displaystyle\int_0^\infty \sin(3t)\delta(t - \frac{\pi}{2})\,dt$

25. Evaluate, using Laplace transforms: $\sin(t) * t$

26. Characterize ALL solutions to $y'' + 4y = u_1(t-1)$, $y(0) = 1$, $y'(0) = 1$.

27. Define $\delta(t-c)$ as a limit of "regular" functions.

28. If $y'(t) = \delta(t-c)$, what is $y(t)$?

29. Show that $\mathcal{L}(g(t+c)) = e^{cs}(G(s) - \int_0^c e^{-st}g(t)\,dt)$. This might be useful, as mentioned in the notes on inverting $f(t)u_c(t)$.

30. What was the *ansatz* we used to obtain the characteristic equation?

31. For the following differential equations, (i) Give the general solution (all possible solutions), (ii) Solve for the specific solution, if its an IVP, (iii) State the interval for which the solution is valid.

(a) $y' = 2\cos(3x) \qquad y(0) = 2$

(b) $y' - 0.5y = 0 \qquad y(0) = 200$

(c) $y' - 0.5y = e^{2t} \qquad y(0) = 1$

(d) $y'' + 4y' + 5y = 0, \quad y(0) = 1, y'(0) = 0$

(e) $y' = 1 + y^2$

(f) $y' = \frac{1}{2}y(3-y)$

(g) $\sin(2x)\mathrm{dx} + \cos(3y)\mathrm{dy} = 0$

(h) $y'' + 2y' + y = 2e^{-t}, \quad y(0) = 0, y'(0) = 1$

(i) $y' = xy^2$

(j) $2xy^2 + 2y + (2x^2y + 2x)y' = 0$

(k) $9y'' - 12y' + 4y = 0, y(0) = 0, y'(0) = -2$

(l) $y'' + 4y = t^2 + 3e^t, y(0) = 0, y'(0) = 1.$

32. For more practice in using the Method of Undetermined Coefficients, look at problems 19-26, p. 171 (all solutions are in the back of the book).

33. Suppose $y' = -ky(y-1)$, with $k > 0$. Sketch the phase diagram. Find and classify the equilibrium. Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down. Finally, get the analytic (general) solution.

34. Let $my'' + \gamma y' + ky = F\cos(wt)$. What are the conditions on m, γ, k to guarantee that the solutions exhibit beating? resonance?

35. Let $y' = 2y^2 + xy^2$, $y(0) = 1$. Solve, and find the minimum of y. Hint: Determine the interval for which the solution is valid.

36. Problem 15, p. 199 (Done as a group quiz)

37. Solve, and determine how the solution depends on the initial condition, $y(0) = y_0$: $y' = 2ty^2$

38. Problem 7, p. 190