General Review Solutions

This sheet also includes problems from the previous two review sheets (Ch. 1-3 and Laplace)

1. Solve (use any method if not otherwise specified):

   (a) $(2x-3x^2)\frac{dx}{dt} = t\cos(t)$ (Seperable)

   $$\int 2x - 3x^2\ dx = \int t\cos(t)\ dt$$

   Integrate the right-hand side of the equation by parts, using a table:

   $$x^2 - x^3 = t\sin(t) + \cos(t) + C$$

   (b) $y'' + 2y' + y = \sin(3x)$ (Undetermined Coefficients) First, homogeneous part of the solution:

   $$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0$$

   so that $y_h = c_1\mathrm{e}^{-t} + c_2t\mathrm{e}^{-t}$. Now the particular solution:

   $$y_p = A\cos(3t) + B\sin(3t)$$

   Substitution into the diff. eqn. will give $A = -\frac{6}{100} = -\frac{3}{50}$, $B = -\frac{8}{100} = -\frac{2}{25}$, so:

   $$y = c_1\mathrm{e}^{-t} + c_2t\mathrm{e}^{-t} - \frac{3}{50}\cos(3t) - \frac{2}{25}\sin(3t)$$

   (c) $(x^2 + xy)y' = x^2 + y^2$ (Homogeneous)

   $$v + x\frac{dv}{dx} = \frac{1+v^2}{1+v}$$

   and note that, performing long division yields

   $$\int \frac{1+v}{1-v}\ dv = \int 1 - \frac{2}{1-v}\ dv \quad \Rightarrow \quad \frac{y}{x} - 2\ln|1 - \frac{y}{x}| = \ln|x| + C$$

   (d) $y'' - 3y' + 2y = \mathrm{e}^{2t}$ (Undetermined Coeffs) We see that $r = 2, 1$, so $y_h = c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{t}$. Our initial guess for y_p would be $A\mathrm{e}^{2t}$, but its in the homogeneous part. Therefore, $y_p = At\mathrm{e}^{2t}$, and:

   $$y = c_1\mathrm{e}^{2t} + c_2\mathrm{e}^{t} + t\mathrm{e}^{2t}$$

   (e) $xy' = y + x\cos^2\left(\frac{y}{x}\right)$ (Homogeneous) Substituting and simplifying, we get:

   $$\int \sec^2(v)\ dv = \int \frac{1}{x}\ dx \quad \Rightarrow \quad \tan(\frac{y}{x}) = \ln|x| + C$$

   (f) $x' = \sqrt{t}\mathrm{e}^{-t} - x$ (Linear- Integ. Factor)

   $$(x\mathrm{e}^t)' = t^{1/2} \quad \Rightarrow \quad x = \frac{2}{3}t^{3/2}\mathrm{e}^{-t} + C\mathrm{e}^{-t}$$

   (g) $y'' - xy' - 2y = 0$ (Power series, assume $x_0 = 0$). Take $y = \sum a_n x^n$ and substitute into the differential equation. After looking at where each power series begins, start each sum with x^1:

   $$-2y = -2a_0 + \sum_{n=1}^{\infty} -2a_n x^n \qquad -xy' = \sum n = 1^\infty - a_n n x^n \quad y'' = 2a_2 + \sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)x^n$$

   Now we get that $a_2 = a_0$, and the recursion relation:

   $$a_{n+2} = \frac{1}{n+1}a_n,\ n = 1, 2, \ldots$$

   so that:

   $$y(x) = a_0 + a_1x + a_0x^2 + \frac{1}{2}a_1x^3 + \frac{1}{3}a_0x^4 + \frac{1}{8}a_1x^5 + \frac{1}{15}a_0x^6 + \ldots$$

(h) $x' = 2 + 2t^2 + x + t^2 x$ (Linear- Integ. Factor)

$$(x e^{-(t+(1/3)t^3)})' = 2(1+t^2)e^{-(t+(1/3)t^3)}$$

(use a "u, du" substitution to integrate), so

$$x = -2 + Ce^{t+(1/3)t^3}$$

2. Obtain the general solution in terms of α, then determine a value of α so that $y(t) \to 0$ as $t \to \infty$:

$$y'' - y' - 6y = 0, \quad y(0) = 1, y'(0) = \alpha$$

The general solution is:

$$y = \left(\frac{2+\alpha}{5}\right) e^{3t} + \left(\frac{3-\alpha}{5}\right) e^{-2t}$$

So, $\alpha = -2$.

3. The Wronskian of two functions is $W(t) = t^2 - 4$. Are the solutions linearly independent? Why or why not? As functions, they are linearly independent on any interval not containing 0. As solutions to a differential equation, the interval must be $t > 0$ or $t < 0$, and then the two functions could be linearly independent solutions.

4. Compute $\mathcal{L}(t\cos(t))$ by using the definition of the Laplace transform. Use Euler's formula:

$$te^{it} = t\cos(t) + it\sin(t) \qquad \text{so that} \qquad \mathcal{L}(te^{it}) = \mathcal{L}(t\cos(t)) + i\mathcal{L}(t\sin(t))$$

Now, $\mathcal{L}(te^{it})$:

$$\int_0^\infty te^{-(s-i)t}\, dt = \left.\frac{-t}{(s-i)}e^{-(s-i)t}\right|_0^\infty - \left.\frac{1}{(s-i)^2}e^{-(s-i)t}\right|_0^\infty = \frac{1}{(s-i)^2} = \frac{s^2 - 1 + 2is}{(s^2+1)^2}$$

so the answer is: $(s^2-1)/(s^2+1)^2$

5. Write 2^i and $\frac{1-3i}{2+i}$ in $a + bi$ form.

$$2^i = e^{\ln(2^i)} = e^{i\ln(2)} = \cos(\ln(2)) + i\sin(\ln(2))$$

$$\frac{(1-3i)(2-i)}{5} = \frac{-1}{5} - \frac{7}{5}i$$

6. Let $\mathbf{x}' = A\mathbf{x}$, where A is given below. Give a complete analysis of each, including (1) Stability classification (Poincaré), (2) Analytic solution, (3) Fundamental Matrix, (4) The Matrix Exponential (leave in factored form), and (5) Sketch the direction field.

(a) $\begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$ $\quad$ $\mathrm{Tr}(A) = 0 \quad \lambda_1 = i$, $\det(A) = 1 \quad \mathbf{v} = (5, 2-i)^T$, $\Delta = -4$ $\quad$ CENTER

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix} + c_2 \begin{bmatrix} 5\sin(t) \\ 2\sin(t) - \cos(t) \end{bmatrix}$$

$$\Psi(t) = \begin{bmatrix} 5\cos(t) & 5\sin(t) \\ 2\cos(t) + \sin(t) & 2\sin(t) - \cos(t) \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 5 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} (-\frac{1}{5}) \begin{bmatrix} -1 & 0 \\ -2 & 5 \end{bmatrix}$$

(b) $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ $\quad$ $\mathrm{Tr}(A) = 2$ $\quad \lambda = 1$ (double) $\quad$ Degenerate Node

$\det(A) = 1$ $\quad \mathbf{v} = (2,1)^T$

$\Delta = 0$ $\quad \mathbf{q} = (1,0)^T$

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^t \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$\Psi(t) = \begin{bmatrix} 2e^t & 2te^t + 1 \\ e^t & te^t \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} (-1) \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

(c) $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ $\quad$ $\mathrm{Tr}(A) = 1$ $\quad \lambda = 2, -1$ $\quad$ SADDLE

$\det(A) = -2$ $\quad \mathbf{v}_1 = (2,1)^T$

$\Delta = 9$ $\quad \mathbf{v}_2 = (1,2)^T$

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Psi(t) = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} (\frac{1}{3}) \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

(d) $\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$ $\quad$ $\mathrm{Tr}(A) = -2$ $\quad \lambda_1 = -1 + 2i$ $\quad$ SPIRAL SINK

$\det(A) = 5$ $\quad \mathbf{v} = (2i, 1)^T$

$\Delta = -16$

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}$$

$$\Psi(t) = e^{-t} \begin{bmatrix} -2\sin(2t) & 2\cos(2t) \\ \cos(2t) & \cos(2t) \end{bmatrix}$$

$$e^{At} = e^{-t} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix} (-\frac{1}{2}) \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$$

(e) $\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$ $\quad$ $\mathrm{Tr}(A) = -2$ $\quad \lambda = 0, -2$ $\quad$ Line of Fixed Points

$\det(A) = 0$ $\quad \mathbf{v}_1 = (3,4)^T$

$\Delta = 4$ $\quad \mathbf{v}_2 = (1,2)^T$

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Psi(t) = \begin{bmatrix} 3 & e^{-2t} \\ 4 & 2e^{-2t} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 3 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0e^{-2t} & \end{bmatrix} (\frac{1}{2}) \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

7. Let $y''' - y' = te^{-t} + 2\cos(t)$. Determine a suitable form for the particular solution, y_p. Do not solve for the coeffs.

$$r^3 - r = 0 \Rightarrow r = 0, r = \pm 1$$

so $y_h = c_1 + c_2 e^t + c_3 e^{-t}$. Our first guess would be: $y_p = (A + Bt)e^{-t}$, but Ae^{-t} is already part of y_h. Therefore, guess that

$$y_{p_1} = t(A_1 + B_1 t)e^{-t} \quad y_{p_2} = A_2\cos(t) + B_2\sin(t)$$

8. Write the differential equation associated with *Resonance* and *Beating*. Discuss under what conditions we can expect each type of behavior.

$$u'' + \omega^2 u = F_0 \cos(\alpha t)$$

The differential equation has no damping term, and has a periodic forcing function. If $\omega = \alpha$, we get RESONANCE. If $\omega \neq \alpha$, then BEATING.

9. Problem 6, p. 369. (From HW, see back of book).

10. Suppose that we have a mass-spring system modelled by the differential equation

$$x'' + 2x' + x = 0, x(0) = 2, x'(0) = -3$$

Find the solution, and determine whether the mass ever crosses $x = 0$. If it does, determine the velocity at that instant. See if it crosses if the velocity is cut in half.

$$x(t) = \mathrm{e}^{-t}(2 - t)$$

so that the velocity at $t = 2$ is $-\mathrm{e}^{-2}$. If the initial velocity is cut in half,

$$x(t) = \mathrm{e}^{-t}(2 + \frac{t}{2})$$

which doesn't cross the $t-$axis in positive time.

11. How is it possible to construct a fundamental set of solutions to $\mathbf{x}' = A(t)\mathbf{x}$ if we only have a computer program that solves an initial value problem?

Solve the initial value problems: $\mathbf{x}' = A(t)\mathbf{x}$, $\mathbf{x}(0) = \mathbf{e}_i$ (See Theorem 7.4.4, p. 368)

12. For problems 5-14, p. 478, determine the equilibria and classify stability based on the Poincaré diagram. See the back of the book.

13. Let $y(x)$ be a power series solution to $(1 - x)y'' + y = 0,\ x_0 = 0$. Find the recurrence relation, and write out the first 6 terms of y.

$$a_n = \frac{n-2}{n} a_{n-1} + \frac{1}{n(n-1)} a_{n-2}$$

The first six terms are listed below:

$$a_0, a_1, -\frac{1}{2}a_0, -\left(\frac{a_0 + a_1}{6}\right), -\left(\frac{a_0 + 2a_1}{24}\right), -\left(\frac{a_0}{60} + \frac{a_1}{24}\right), -\left(\frac{7a_0}{720} + \frac{a_1}{40}\right)$$

14. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0,\ x_0 = 1$. Find the recurrence relation and write out the first 6 terms of y.

$$a_n = \frac{1}{n} a_{n-2}, n = 2, 3, \ldots$$

The first six terms are:

$$a_0, a_1, \frac{1}{2}a_0, \frac{1}{3}a_1, \frac{1}{8}a_0, \frac{1}{15}a_1, \frac{1}{48}a_0$$

15. True or False: If

$$dx/dt = F(x, y), \quad dy/dt = G(x, y)$$

and (x^*, y^*) is a critical point, then any other solution cannot reach the critical point in finite time. (Be sure to explain why). True. Solutions cannot cross in the phase diagram (by the existence and uniqueness theorem).

16. Let $x' = \sin(y)$, $y' = \sin(x)$ Find all equilibria, and classify the stability. The equilibria are are $y = k\pi$, $x = k\pi$, and

$$Df = \begin{bmatrix} 0 & \cos(y) \\ \cos(x) & 0 \end{bmatrix}$$

so we have 4 choices:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \pi + 2\pi k \\ \pi + 2\pi k \end{bmatrix}, \begin{bmatrix} \pi + 2\pi k \\ 2\pi k \end{bmatrix}, \begin{bmatrix} 2\pi k \\ 2 + 2\pi k \end{bmatrix}, \begin{bmatrix} 2\pi k \\ 2\pi k \end{bmatrix}$$

For which Df is (respectively):

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

In all cases, $\text{Tr}(A) = 0$- The determinants are $1, -1, -1, 1$ (respectively), and Δ is $-4, 4, 4, -4$, respectively. Therefore, we would conclude that the linear analysis says that equilibria of the first and third types are saddles and the others are centers.

17. p. 502, 1-6. Try a couple of these examples of Competing Species. (See back of book)

18. p. 503, 12(a-d). (Last HW set)

19. Analyze how the origin changes classification with respect to α if:

$$\mathbf{x}' = \begin{pmatrix} 1 & \alpha \\ -\alpha & -2 \end{pmatrix} \mathbf{x}$$

Done in Class.

20-29. See the Laplace Review Sheet.

30. What was the *ansatz* we used to obtain the characteristic equation? $y(t) = \mathrm{e}^{rt}$

31. For the following differential equations, (i) Give the general solution (all possible solutions), (ii) Solve for the specific solution, if its an IVP, (iii) State the interval for which the solution is valid.

    (a) $y' = 2\cos(3x), y(0) = 2$: $y(x) = \frac{2}{3}\sin(3x) + C$, $y(x) = \frac{2}{3}\sin(3x) + 2$, Valid for all x.

    (b) $y' - 0.5y = 0, y(0) = 200$: $y(x) = 0$ and $y(x) = A\mathrm{e}^{0.5x}$, $y(x) = 200\mathrm{e}^{0.5x}$, Valid for all x.

    (c) $y' - 0.5y = \mathrm{e}^{2t}, y(0) = 1$: $y(t) = A\mathrm{e}^{0.5t} + \frac{2}{3}\mathrm{e}^{2t}$, $y(t) = \frac{1}{3}\mathrm{e}^{0.5t} + \frac{2}{3}\mathrm{e}^{2t}$, Valid for all t.

    (d) $y'' + 4y' + 5y = 0, y(0) = 1, y'(0) = 0$: $y(x) = 0$ and $y(x) = \mathrm{e}^{-2x}(c_1\cos(x) + c_2\sin(x))$, $y(x) = \mathrm{e}^{-2x}(\cos(x) + 2\sin(x))$, Valid for all x.

    (e) $y' = 1 + y^2$ $\tan^{-1}(y) = x + C$, so $y = \tan(x + C)$. Valid in a small interval about any x_0.

    (f) $y' = \frac{1}{2}y(3 - y)$ Solve by seperation of variables and partial fractions. $y(x) = 0$ and $y(x) = 3$ are two special (equilibrium) solutions. The general solution is $y = \frac{3A\mathrm{e}^{\frac{3}{2}x}}{1+3A\mathrm{e}^{\frac{3}{2}x}}$, which is valid for all x (after we solve, we can check).

    (g) $\sin(2x)\mathrm{d}x + \cos(3y)\mathrm{d}y = 0$ $\frac{1}{3}\sin(3y) = \frac{1}{2}\cos(2x) + C$, using the existence and uniqueness theorem, we can't have $3y$ be an odd multiple of $\frac{\pi}{2}$.

    (h) $y'' + 2y' + y = 2\mathrm{e}^{-t}, \quad y(0) = 0, y'(0) = 1$ The general solution: $\mathrm{e}^{-t}(c_1 + c + 2t + t^2)$, and the specific solution is: $\mathrm{e}^{-t}(t + t^2)$, Valid for all t.

    (i) $y' = xy^2$ General solution is $y = \frac{-2}{x^2+C}$. Note that the interval depends on the initial conditions!

    (j) $2xy^2 + 2y + (2x^2y + 2x)y' = 0$ This equation is **EXACT**. General solution: $x^2y^2 + 2xy = C$. The solution exists for $x \neq 0$ and $xy \neq -1$. (Otherwise we get vertical y').

    (k) $9y'' - 12y' + 4y = 0, y(0) = 0, y'(0) = -2$ The solution is valid for all time. The specific solution is: $y(t) = -2t\mathrm{e}^{\frac{2}{3}t}$

(l) $y'' + 4y = t^2 + 3e^t, y(0) = 0, y'(0) = 1$. The solution is valid for all time. The solution is:

$$y(t) = \frac{1}{5}\sin(2t) - \frac{19}{4}\cos(2t) + \frac{3}{5}e^t + \frac{1}{4}t^2 - \frac{1}{8}$$

32. For more practice in using the Method of Undetermined Coefficients, look at problems 19-26, p. 171 (all solutions are in the back of the book).

33. Suppose $y' = -ky(y-1)$, with $k > 0$. Sketch the phase diagram. Find and classify the equilibrium. Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down. Finally, get the analytic (general) solution.

    The analytic solution is (solve for y):
    $$\frac{y-1}{y} = Ae^{kt}$$

    The equilibria are always at $y = 0$ (Unstable) and $y = 1$ (Stable). From the phase diagram, if $y < 0$, then y is decreasing and concave down. If $0 < y < \frac{1}{2}$, y is increasing and concave up. If $\frac{1}{2} < y < 1$, y is increasing and concave down. If $y > 1$, y is decreasing and concave up.

34. Let $my'' + \gamma y' + ky = F\cos(wt)$. What are the conditions on m, γ, k to guarantee that the solutions exhibit beating? resonance? See Problem 8 (Sorry about doubling the question up).

35. Let $y' = 2y^2 + xy^2$, $y(0) = 1$. Solve, and find the minimum of y. Hint: Determine the interval for which the solution is valid.

    The solution is (seperable equation):
    $$y = \frac{-1}{\frac{1}{2}x^2 + 2x - 1}$$

    From the diff. eqn, $y' = 0$ where $x = -2$ or $y = 0$. We see that $y \neq 0$, so a candidate for the minimum is $x = -2$. We also can check that y has vertical asymptotes for $x = -2 \pm \sqrt{6}$. The minimum occurs at $x = -2$.

36. Problem 15, p. 199 (Done as a group quiz) We treat this problem as three seperate problems.

    First, $u'' + u = F_0 t$, $u(0) = 0$, $u'(0) = 0$. The homogeneous part is $u_h = c_1\cos(t) + c_2\sin(t)$ Undetermined coefficients gives $u_p = F_0 t$, so the solution is:

    $$u(t) = -F_0\sin(t) + F_0 t, \quad 0 \le t \le \pi$$

    At $t = \pi$, we want to keep u continuous, so we have: $u'' + u = F_0(2\pi - t)$, $u(\pi) = F_0\pi$, $u'(\pi) = 2F_0$. Solving this IVP yields:
    $$u(t) = -3F_0\sin(t) - F_0 t + 2\pi F_0$$

    Finally, the last IVP has: $u'' + u = 0, \quad u(2\pi) = 0, u'(2\pi) = -4F_0$, so that:

    $$u(t) = -4F_0\sin(t), \quad t > 2\pi$$

    (You can double check this solution by writing $F(t)$ using the Heaviside function, then do Laplace).

37. Solve, and determine how the solution depends on the initial condition, $y(0) = y_0$: $y' = 2ty^2$

    The solution is: $y(t) = \frac{-1}{t^2 - \frac{1}{y_0}}$

38. Problem 7, p. 190 See the solution in the back of the book.