Linear Operators

In this set of notes, we see how the definition of a *linear* differential equation comes into being.

1. Recall the following definition from linear algebra:

   Let V, W be vector spaces, and let $T : V \to W$. Then T is a linear function if:

   - $T(x + y) = T(x) + T(y)$, for all $x, y \in V$
   - $T(cx) = cT(x)$, for all $x \in V$, and $c \in \mathbb{R}$

2. Definition: $C^n[a, b]$ is the space of all $n-$times differentiable functions over the interval from a to b.

3. In linear algebra, we saw that $C^n[a, b]$ is a vector space (it satisfies the 10 axioms), and a "vector" in the space is actually a function, $f(x)$.

4. Definition: A function whose domain is a space of functions is called an *operator*. For example, if $T(f(x)) = f'(x)$, then T is called the differentiation operator.

5. Other examples of operators T:

   - $T(f) = \int_0^t f(s)\, ds$
   - $T(f) = f'(x) + 3f(x)$
   - $T(f) = (f'(x))^2 + f(x)$

6. Exercise: Show that the first two operators above are linear operators.

7. Obtaining the operator associated with a differential equation.

   Examples:

   - If $y' = \frac{1-t}{1+y}$, then $T(y) = y' + yy'$
   - If $y' + t^2 y = \sin(t)$, then $T(y) = y' + t^2 y$
   - If $y'' + 3y' + 2y - 5 = 0$, then $T(y) = y'' + 3y' + 2y$

   In general, to obtain the operator associated with a differential equation, we first move all of the terms involving the function to the left side, and all of the other terms on the right. The left side now describes the operator associated to a differential equation.

8. Definition. Let $y' = f(t, y)$ be a first order differential equation. Let $T(y)$ be the associated operator, so that:
   $$y' = f(t, y) \Rightarrow T(y) = g(t)$$
   The the differential equation is said to be linear if T is a linear operator.

9. Solving a linear equation. Recall from linear algebra that, if $Ax = b$, then the solution can be written as:
   $$x = x_h + x_p$$
   where x_h solves $Ax = 0$ and x_p is a particular solution to $Ax = b$. The same thing occurs for a linear differential equation: If $y' + p(t)y = g(t)$, then the solution can be written as:
   $$y = y_h + y_p$$
   where y_h solves $T(y) = y' + p(t)y = 0$, and y_p is a particular solution to $T(y) = g(t)$.

10. Example: $y' + \frac{2}{t} y = 4t$. The operator is $T(y) = y' + \frac{2}{t} y$, which is linear in y. The solution, by using an integrating factor, is:
    $$y = t^2 + \frac{c}{t^2}$$
    Verify that, if $y = t^2$, then $T(y) = 4t$ (so that this is the particular part of the solution), and if $y = \frac{c}{t^2}$, then $T(y) = 0$ (the homogeneous part of the solution).

11. We will explore these relationships more in Chapter 3.