Partial Fraction Decomposition

Partial Fraction Decomposition is used when we have a fraction, $P(x)/Q(x)$, where P, Q are polynomials, and the degree of P is less than the degree of Q[1]

Assume Q is fully factored. We have 4 cases that we will consider.

Case I : Q has distinct linear factors,

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\dots(a_kx + b_k)$$

Then:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_k}{a_kx + b_k}$$

Case II : Q has some repeated linear factors. Let $a_1x + b_1$ be repeated r times. Then, instead of the single term $A_1/(a_1x + b_1)$, we have:

$$\frac{B_1}{a_1x + b_1} + \frac{B_2}{(a_1x + b_1)^2} + \dots + \frac{B_r}{(a_1x + b_1)^r}$$

Case III : Q has some irreducible quadratic factors, not repeated. Let $ax^2 + bx + c$ be an irreducible quadratic factor for Q. Then the decomposition will have the term:

$$\frac{Ax + B}{ax^2 + bx + c}$$

Case IV : Q has some irreducible quadratic factors, some repeated. Suppose that $ax^2 + bx + c$ is a repeated quadratic factor (repeated r times). Then, instead of the single expression in Case III, we will have:

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

Worked Examples and Exercises

1. Worked Example:

$$\frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}$$

Solution:

First, factor the denominator: $x(2x - 1)(x + 2)$. We have distinct linear factors. Now we try to solve the following for A, B, and C:

$$\frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2} = \frac{A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)}{x(2x - 1)(x + 2)}$$

Now, simplify the numerator, and we can see that we are solving:

$$x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$$

which is really three equations, where we set the coefficients of x^2 equal, coefficients of x equal, and constants equal:

$$1 = 2A + B + 2C, 2 = 3A + 2B - C, -1 = -2A$$

Solving, we get: $A = 1/2, B = 1/5$ and $C = -1/10$.

[1] If the degree of $P > Q$, then perform long division first.

2. Worked Example:

$$\frac{x^2-2}{x(x^2+2)}$$

Expanding:

$$\frac{x^2-2}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$$

Getting a common denominator,

$$x^2-2 = A(x^2+2) + (Bx+C)x = (A+B)x^2 + Cx + 2A$$

so that:

$$1 = A+B, 0 = C, 2A = -2$$

so: $A=-1, B=2$ and $C=0$:

$$\frac{x^2-2}{x(x^2+2)} = \frac{-1}{x} + \frac{2x}{x^2+2}$$

The solutions to the following questions are on my web site:

3. $\dfrac{2x+3}{(x+1)^2}$

4. $\dfrac{4x^2-7x-12}{x(x+2)(x-3)}$

5. $\dfrac{x^2+3}{x^3+2x}$

6. $\dfrac{x^2-2x-1}{(x-1)^2(x^2+1)}$

7. $\dfrac{2x^3-x^2+3x-1}{(x^2+1)(x^2+2)}$

8. $\dfrac{x^4+1}{x(x^2+1)^2}$