Differential Equations with Discontinuous Forcing

The following theorem is not in the text, but provides an insight into what kinds of solutions to expect:

Theorem: Let $y'' + p(t)y' + g(t)y = f(t)$, $y(t_0) = y_0$, $y'(t_0) = v_0$. Let f be continuous at t_0 and piecewise continuous on the interval $[a, b]$. Let p, g be continuous on the interval.

If: y, y' are chosen to be continuous at every point in the interval, then there exists a unique solution on the interval. In this case, y'' is piecewise continuous.

An example:

$$y'' + 4y = u_1(t)\cos(t-1), \quad y(0) = 1, y'(0) = 2$$

Notice that the forcing function has a jump discontinuity at $t = 1$.

We solve using Laplace Transforms:

$$s^2Y - s - 2 + 4Y = \mathrm{e}^{-s}\frac{s}{s^2+1}$$

Solve for Y:

$$Y = \mathrm{e}^{-s}\frac{s}{(s^2+1)(s^2+4)} + \frac{s+2}{s^2+4}$$

and invert to get that:

$$y(t) = \cos(2t) + \sin(2t) + \frac{1}{3}u_1(t)\left(\cos(t-1) - \cos(2t-2)\right)$$

Which we can analyze better as:

$$y(t) = \begin{cases} \cos(2t) + \sin(2t), & 0 \le t \le 1 \\ \cos(2t) + \sin(2t) + \frac{1}{3}(\cos(t-1) - \cos(2t-2)), & t > 1 \end{cases}$$

We see that $y(1)^- = \cos(2) + \sin(2)$, and $y(1)^+ = \cos(2) + \sin(2) + \frac{1}{3}(1-1)$, so y is continuous at $t = 1$.

$$y'(t) = \begin{cases} 2(-\sin(2t) + \cos(2t)), & 0 \le t < 1 \\ 2(-\sin(2t) + \cos(2t)) + \frac{1}{3}(-\sin(t-1) + 2\sin(2t-2)), & t > 1 \end{cases}$$

Where we see that the right and left limits of y' at $t = 1$ are equal. We can therefore change the inequality $0 \le t < 1$ to $0 \le t \le 1$, and examine y'':

$$y''(t) = \begin{cases} 4(-\cos(2t) - \sin(2t)), & 0 \le t < 1 \\ 4(-\cos(2t) - \sin(2t)) + \frac{1}{3}(-\cos(t-1) + 4\cos(2t-2)), & t > 1 \end{cases}$$

Now we see a break at $t = 1$.

Let us examine what the theorem means when it says that we have to *choose* y and y' to be continuous. Let $y'' + 4y = 1 - u_1(t)$, $y(0) = 0$ and $y'(0) = 0$. We solve this using the method of undetermined coefficients.

The homogeneous part of the solution is $c_1\cos(2t) + c_2\sin(2t)$. With the guess that $y_p(t) = A$, we see that $A = \frac{1}{4}$. So the general solution is:

$$y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{4}$$

Using the initial conditions, we get that $c_1 = -\frac{1}{4}$ and $c_2 = 0$. We have now solved the differential equation for $0 \le t \le 1$. You can verify that:

$$y(t) = \begin{cases} -\frac{1}{4}\cos(2t) + \frac{1}{4}, & 0 \le t \le 1 \\ c_1\cos(2t) + c_2\sin(2t), & t > 1 \end{cases}$$

solves the differential equation for any choice of c_1 and c_2. However, the theorem says that we can choose y and y' to be continuous, and in that case, solve for particular values of c_1 and c_2 that uniquely solve the differential equation:

$$y(1) = \frac{1}{4} - \frac{1}{4}\cos(2), \;\; y'(1) = \frac{1}{2}\sin(2)$$

so that:

$$\begin{bmatrix} \cos(2) & \sin(2) \\ -2\sin(2) & 2\cos(2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} - \frac{1}{4}\cos(2) \\ \frac{1}{2}\sin(2) \end{bmatrix}$$