STUDY GUIDE: Ch. 1-3

Boyce and DiPrima, Spring 2000

This review is organized into four main areas: Analysis, Methods, Theory, and Models.

We've seen that a differential equation defines a family of functions (an initial value problem defines a specific function). We've reviewed and extended the material in Calculus IV to include theory and methods to solve first order differential equations of certain types, and to solve second order linear differential equations (with constant coefficients). We've gone further into the theory to discuss when solutions exist, and in particular (3.2/3.3) the structure of solutions to second order linear differential equations.

1 Analysis of Solutions

1. Construct a direction field: Since $y' = f(t, y)$, at each value of (t, y), we can compute the local slope, $y'(t)$.

2. Characterization of special direction fields:

Slopes constant along:	Corresponds to:
Vertical Lines	$y' = f(x)$
Horizontal Lines	$y' = f(y)$
Along $y = mx$	$y' = f(y/x)$

3. The Phase Diagram, and the Direction Field: Given that $y' = f(y)$, we can plot y' vs. y. This gives us information that we can translate to the direction field, a plot of y vs x. This information is summarized in the table below. NOTE: df stands for $\frac{df}{dy}$.

In Phase Diagram:	In Direction Field:
y intercepts	Equilibrium Solutions
$+$ to $-$ crossing	Stable Equilibrium
$-$ to $+$ crossing	Unstable Equilibrium
$y' > 0$	y increasing
$y' > 0$ with max	y is increasing fastest
$y' < 0$ with min	y is decreasing fastest
$y' < 0$	y decreasing
y' and df same sign	y is concave up
y' and df mixed	y is concave down

4. Analysis of $y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$. We saw that, to guarantee that $|y_h| \to \infty$, both exponents need to be positive, and if $|y_h| \to \infty$, both exponents must be negative. If the exponents are mixed in sign, it is possible to choose combinations of c_1 and c_2 to force either $|y_h|$ to infinity or zero.

5. Analysis of $c_1 \cos(wt) + c_2 \sin(wt)$. We saw that we could write this as a single periodic function, $R\cos(wt - \delta)$, with the following conversions:

$$R = \sqrt{c_1^2 + c_2^2}, \qquad \delta = \mathrm{Tan}^{-1}(c_2/c_1)$$

where Tan^{-1} is the four quadrant inverse (versus $\tan^{-1}$, which returns values in $(\pi/2, -\pi/2)$).

6. Analysis of $\frac{F}{\mu^2 - w^2}(\cos(\mu t) - \cos(wt))$ with $\mu \neq w$ and as $\mu \to w$. See the section on oscillator models.

2 Methods

1. First Order

Type:	Method:
$y' = f(x)g(y)$ $y' = f(x)$ $y' = g(y)$	Seperation of variables
$y' + p(t)y = f(t)$	Integrating factor
$y' = f(y/x)$	Homogeneous use $v = y/x$
$M(x, y) + N(x, y)\frac{dy}{dx}$	Exact: $M_y = N_x$

2. Second Order

Let $ay'' + by' + cy = f(t)$. Then $y(t) = y_h(t) + y_p(t)$. We solve for $y_h(t)$ by using the characteristic equation, $ar^2 + br + c = 0$. The solution then depends on the roots:

Distinct Reals:	$c_1 e^{r_1 t} + c_2 e^{r_2 t}$
One Real:	$e^{rt}(c_1 + c_2 t)$
Complex, $r = s \pm iw$	$e^{st}(c_1 \cos(wt) + c_2 \sin(wt))$

We have one method to solve for $y_p(t)$: **The method of undetermined coefficients**.

DISCUSSION: This methods works because if we apply the linear differential operator to polynomials, exponentials, and/or sines/cosines, the output is a polynomial, exponential, and/or sines and cosines. Therefore, we guess that y_p is of the same type as $f(t)$.

CAUTION: If $f(t)$ appears in the homogeneous part of the solution, multiply the ansatz by t until it doesn't. Also, if $f(t)$ is a sum of functions, we can solve for each seperately.

3 Theory

1. **Existence and Uniqueness**

   (a) Linear:

   $y'+p(t)y = f(t)$ at (t_0, y_0): If p, f are continuous on an interval I that contains t_0, then there exists a unique solution to the initial value problem, (whose derivative is continuous), and that solution persists on the full interval I.

   $y''+p(t)y'+q(t)y = f(t)$ at (t_0, y_0): If p, q, f are continuous on an interval I containing t_0, then there exists a unique solution to the intial value problem, (whose second derivative is continuous), and that solution persists on the full interval I.

   (b) Nonlinear: $y' = f(t, y)$, (t_0, y_0):

   i. If f is continuous on a small rectangle containing (t_0, y_0), then there exists a solution to the initial value problem.

   ii. If $\partial f/\partial y$ is continuous on that small rectangle containing (t_0, y_0), then that solution is unique.

   iii. We can only guarantee that the solution persists on a small interval about (t_0, y_0). To find the full interval, we need to actually solve the initial value problem.

2. Linear Operators[1], Linear Independence, Wronskian:

   CONTEXT: n^{th} order linear homogeneous differential equations, like $y' + p(t)y = 0$ and $y'' + p(t)y' + q(t)y = 0$.

   GOAL: Know what it means to have linearly independent functions (to define a fundamental set), and to know how many functions are necessary to form a fundamental set. Understand the role of the Wronskian and Abel's Theorem in this process.

   (a) Definition: A set of functions is linearly independent if the only solution (on the full interval I) to:

   $$c_1y_1(t) + c_2y_2(t) + \ldots + c_ky_k(t) = 0$$

   is the trivial solution:

   $c_1 = c_2 = \ldots = c_k = 0$.

   (b) From the definition, we can construct a set of k equations in k unknowns to form the system of equations:

   $$\begin{aligned} c_1y_1(t) + c_2y_2(t) + \ldots + c_ky_k(t) &= 0 \\ c_1y_1'(t) + c_2y_2'(t) + \ldots + c_ky_k'(t) &= 0 \\ \cdots \quad & \cdots \\ c_1y_1^{(k-1)}(t) + c_2y_2^{(k-1)}(t) + \ldots + c_ky_k^{(k-1)}(t) &= 0 \end{aligned}$$

   From which we form a matrix equation:

   $$\begin{bmatrix} y_1(t) & y_2(t) & \ldots & y_k(t) \\ y_1'(t) & y_2'(t) & \ldots & y_k'(t) \\ \vdots & & \ldots & \vdots \\ y_1^{(k-1)}(t) & y_2^{(k-1)}(t) & \ldots & y_k^{(k-1)}(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

   We see that, if this matrix is invertible for any t_0, then there is only the trivial solution. Therefore, the determinant is a key player, and the Wronskian is the determinant of that matrix.

   CONCLUSION: If there is a t_0 in I so that $W(y_1, \ldots, y_k)(t_0) \neq 0$, then the functions are linearly independent.

   CAUTION: It is possible for $W(y_1, y_2) = 0$ for **all** t, and still have linearly independent functions. For example, look at $t^2|t|$ and t^3 on the interval $-1 < t < 1$.

   (c) In class, we proved that the dimension of the nullspace of the linear operator $L(y)$ was n for an n^{th} order linear differential equation. In other words,

   "There are exactly n linearly independent functions to the n^{th} order linear homogeneous differential equation".

   These n functions make up the **fundamental set** (they form a basis).

   (d) Abel's Theorem:

   DISCUSSION: While the definition of linear independence holds for *any* set of functions, we are especially interested in functions that are solutions to a linear homogeneous differential equation. Abel's Theorem provides that connection.

   THEOREM: If y_1, y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$ on the interval I, then $W(y_1, y_2)$ is either identically zero or never zero on the interval I.

4 Models

We discussed many applications of differential equations, including population models, radioactive decay, Newton's Law of Cooling, Tank mixing problems and Oscillators. Below is list of the things out of these sections we want to be able to do:

[1] The idea of a linear operator and the connections to linear algebra were presented mainly to remind you of what we did in linear algebra, and to facilitate your understanding of the new material. It will not appear on the exam

1. Know the basic radioactive decay model and its solution.

2. Construct an autonomous differential equation to model population growth in the standard model and with an environmental equilibrium. Be able to solve these models analytically (usually requires partial fractions) and graphically.

3. Be able to construct the differential equation corresponding to the tank mixing problem. Be able to solve it (it will be a linear first order equation) and analyze it (using a phase diagram, if possible).

4. Given Newton's Law of cooling model, be able to find the coefficients in the model, solve it analytically, and analyze the behavior of the solutions (also using a phase diagram).

5. Oscillators: (Sections 3.8 and 3.9)

   (a) Know the mass-spring model (what do the coefficients represent?)

   (b) Be able to analyse the possible outcomes of the phsyical problem (versus all mathematically possible outcomes).

   (c) Understand why the phenomena of Beating and Resonance appear in solutions. For what types of differential equations can we expect these phenomena? Understand the transition from Beating to Resonance.

   (d) Solve and analyze the second order linear differential equation, both forced and unforced, and discuss what the solution means in the physical setting (mass-spring).

5 Miscellaneous Questions

1. Some conceptual questions:

   (a) What is an n^{th} order differential equation? What's the difference between a linear and nonlinear differential equation?

   (b) What does it mean for a function to be a solution to a differential equation?

   (c) What's the difference between a differential equation and an initial value problem?

   (d) What do we use a direction field for, and when do we use it?

   (e) Does it make sense to draw a direction field for a second order differential equation?

   (f) What do we use a phase diagram for, and when do we use it?

   (g) What is the relationship between the phase diagram and the direction field (for first order equations).

   (h) What is an autonomous differential equation?

   (i) What's a "transient" part of a solution? What's a "steady-state" part of a solution? When can we expect to get them?

   (j) What's the difference between a "steady-state" solution and an equilibrium solution?

   (k) What's the difference between linearly independent functions and linearly independent solutions to a linear differential equation?

   (l) We said earlier that, under certain circumstances, solutions to a $y' = f(t, y)$ cannot cross in the direction field. What are the conditions?

   (m) Note that $y = 0$ and $y = \cos(t) + \sin(t)$ are both solutions to $y'' + y = 0$, which means that there are an infinite number of crossings between two solutions of the differential equation in the direction field. Does this contradict the conclusions to the previous question?

   (n) We know there are exactly two linearly independent solutions to $ay'' + by' + cy = 0$. What were the initial value problems used to prove this? Is this how we normally construct the fundamental set?

   (o) What was the *ansatz* we used to obtain the characteristic equation?

   (p) Using the two theorems we had about Existence and Uniqueness of linear differential equations, state what should be the Existence and Uniqueness Theorem for: $y''' + p(t)y'' + q(t)y' + g(t)y = 0$.

2. Suppose we wish to study the formation of raindrops in the atmosphere. We will take the simplifying assumption that raindrops are approximately spherical, and our model says that the rate of change of the raindrops' volume is proportional to its surface area.

   We therefore have that:

$$v = \frac{4}{3}\pi r^3 \Rightarrow r = \left(\frac{3v}{4\pi}\right)^{1/3}, \quad \text{and SA} = 4\pi r^2$$

   Then:

$$\frac{dv}{dt} = 4\pi r^2 = 4\pi\left(\frac{3v}{4\pi}\right)^{2/3} = kv^{2/3}$$

   With the initial condition $v(0) = 0$, what does the Existence and Uniqueness theorem say?

Find at least two distinct solutions initial value problem.

Show that:

$$y(t) = \begin{cases} (\frac{k}{3}t - c)^3, & t > \frac{3c}{k} \\ 0, & t \leq \frac{3c}{k} \end{cases}$$

is also a solution, for any positive value of c. What physical implications does this have for the "real world"?

3. An extra problem to analyze:

$$x' = \sin\left(\frac{1}{x}\right)$$

with $x(0) = C$, where C is a small positive number.

   (a) What type of differential equation is this?
   (b) Between what two values does $\sin\left(\frac{1}{x}\right)$ oscillate?
   (c) What are the roots of $\sin\left(\frac{1}{x}\right)$? Try to get a general expression for them.
   (d) Describe in words what happens to the $\sin\left(\frac{1}{x}\right)$ as x goes to zero. How about as x goes to infinity?
   (e) The x-intercepts of a curve in the phase diagram have special meaning. What is it?
   (f) Sketch a graph of $\sin\left(\frac{1}{x}\right)$ (by hand, as best you can).
   (g) What, if anything, can you say about the qualitative behavior of the solution satisfying the initial condition $x(0) = C$?

4. For the following differential equations, (i) Give the general solution (all possible solutions), (ii) Solve for the specific solution, if its an IVP, (iii) State the interval for which the solution is valid.

   (a) $y' = 2\cos(3x) \qquad y(0) = 2$
   (b) $y' - 0.5y = 0 \qquad y(0) = 200$
   (c) $y' - 0.5y = \mathrm{e}^{2t} \qquad y(0) = 1$
   (d) $y'' + 4y' + 5y = 0, \quad y(0) = 1, y'(0) = 0$
   (e) $y' = 1 + y^2$
   (f) $y' = \frac{1}{2}y(3 - y)$
   (g) $\sin(2x)\mathrm{dx} + \cos(3y)\mathrm{dy} = 0$
   (h) $y'' + 2y' + y = 2\mathrm{e}^{-t}, \quad y(0) = 0, y'(0) = 1$
   (i) $y' = xy^2$
   (j) $2xy^2 + 2y + (2x^2y + 2x)y' = 0$
   (k) $9y'' - 12y' + 4y = 0, y(0) = 0, y'(0) = -2$
   (l) $y'' + 4y = t^2 + 3\mathrm{e}^t, y(0) = 0, y'(0) = 1.$

5. For more practice in using the Method of Undetermined Coefficients, look at problems 19-26, p. 171 (all solutions are in the back of the book).

6. Suppose $y' = -ky(y - 1)$, with $k > 0$. Sketch the phase diagram. Find and classify the equilibrium. Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down. Finally, get the analytic (general) solution.

7. Let $my'' + \gamma y' + ky = F\cos(wt)$. What are the conditions on m, γ, k to guarantee that the solutions exhibit beating? resonance?

8. Let $y' = 2y^2 + xy^2$, $y(0) = 1$. Solve, and find the minimum of y. Hint: Determine the interval for which the solution is valid.

9. Problem 15, p. 199 (Done as a group quiz)

10. Solve, and determine how the solution depends on the initial condition, $y(0) = y_0$: $y' = 2ty^2$

11. Problem 7, p. 190