FPI with Systems (Exercise Solutions)

1. Let $\boldsymbol{x}_{n+1} = T\boldsymbol{x} + \boldsymbol{c}$. Show that

$$\boldsymbol{x}_n = T^n \boldsymbol{x}_0 + \left(T^{n-1} + T^{n-2} + \cdots + T^2 + T + I\right)\boldsymbol{c}$$

SOLUTION: Go through the first few of them:

$$\begin{aligned}
\boldsymbol{x}_1 &= T\boldsymbol{x}_0 + \boldsymbol{c} \\
\boldsymbol{x}_2 &= T\boldsymbol{x}_1 + \boldsymbol{c} = T(T\boldsymbol{x}_0 + \boldsymbol{c}) + \boldsymbol{c} = T^2\boldsymbol{x}_0 + (T+I)\boldsymbol{c} \\
\boldsymbol{x}_3 &= T\boldsymbol{x}_2 + \boldsymbol{c} = T(T^2\boldsymbol{x}_0 + (T+I)\boldsymbol{c}) + \boldsymbol{c} = T^3\boldsymbol{x}_0 + (T^2 + T + I)\boldsymbol{c}
\end{aligned}$$

and we see the general pattern:

$$\boldsymbol{x}_n = T^n \boldsymbol{x}_0 + \left(T^{n-1} + T^{n-2} + \cdots + T^2 + T + I\right)\boldsymbol{c}$$

2. Show that, if $\boldsymbol{x}_{n+1} = T\boldsymbol{x}_n + \boldsymbol{c}$ and $\boldsymbol{r} = T\boldsymbol{r} + \boldsymbol{c}$, then using an induced matrix norm,

$$\|\boldsymbol{x}_k - \boldsymbol{r}\| \le \|T\|^k \|\boldsymbol{x}_0 - \boldsymbol{r}\|$$

SOLUTION: Easiest to show working up from $k = 1$:

$$\|\boldsymbol{x}_1 - \boldsymbol{r}\| = \|T\boldsymbol{x}_0 + \boldsymbol{c} - T\boldsymbol{r} - \boldsymbol{c}\| = \|T(\boldsymbol{x}_0 - \boldsymbol{r})\| \le \|T\|\,\|\boldsymbol{x}_0 - \boldsymbol{r}\|$$

Similarly,

$$\|\boldsymbol{x}_2 - \boldsymbol{r}\| = \|T\boldsymbol{x}_1 + \boldsymbol{c} - T\boldsymbol{r} - \boldsymbol{c}\| = \|T(\boldsymbol{x}_1 - \boldsymbol{r})\| \le \|T\|\,\|\boldsymbol{x}_1 - \boldsymbol{r}\| \le \|T\|^2 \|\boldsymbol{x}_0 - \boldsymbol{r}\|$$

And one more:

$$\|\boldsymbol{x}_3 - \boldsymbol{r}\| = \|T(\boldsymbol{x}_2 - \boldsymbol{r})\| \le \|T\|^3 \|\boldsymbol{x}_0 - \boldsymbol{r}\|$$

Continuing in this fashion, we get the desired result:

$$\|\boldsymbol{x}_k - \boldsymbol{r}\| \le \|T\|^k \|\boldsymbol{x}_0 - \boldsymbol{r}\|$$

3. Exercise 1(a), 1(c) (only Jacobi). Also see the code online that will split the matrix A for you.