Matlab's Numerical Integration Commands

The relevant commands we consider are `quad` and `dblquad`, `triplequad`. See the Matlab help files for other integration commands. By the way, "quad" refers to "adaptive quadrature".

To integrate: $\int_a^b f(x)\,dx$, we use the command:

```
Q=quad(function, a, b);
Q=quad(function, a, b, tol);
```

The second version of the `quad` function is used if there is a user-defined tolerance. The default tolerance is 10^{-6}.

The only slightly tricky thing here is in how to define and pass in the function $f(x)$. We have several options, depending on the complexity of your definition of $f(x)$. Below we give three examples:

Example:

$$\int_0^2 \frac{1}{2x^3 - 2x - 5}\,dx$$

which in Matlab would be:

```
f1=inline('1./(2*x.^3-2*x-5)');
Q=quad(f1,0,2);
```

We could also define the function using "anonymous handles" (See Matlab documentation for more):

```
f2=@(x)1./(2*x.^3-2*x-5);
Q=quad(f2,0,2);
```

We could also define the function using an M-file. Here is a sample, which we'll save as `myfun.m`:

```
function y=myfun(x)
y=1./(2*x.^3-2*x-5);
```

In the Matlab command window, we would type:

```
Q=quad(@myfun,0,2);
```

Similarly, if you're integrating using a built-in function like:

$$\int_0^5 \sin(x)\,dx$$

you would type: `Q=quad(@sin, 0, 5);`

Double Integrals

Suppose we wish to integrate: $\int_1^2 \int_0^3 x^2 y\,dx\,dy$

As before, we can use either an inline function, an anonymous function handle, or an M-file:

- Using an inline function. When defining a function of more than one variable, we can list the order in which they should appear at the end:

  ```
  f1=inline('x.^2.*y','x','y');
  Q=dblquad(f,0,3,1,2);
  ```

- Using an anonymous function handle:

  ```
  f2=@(x,y)x.^2.*y;
  Q=dblquad(f2,0,3,1,2);
  ```

- Using an M-file. First type and save the M-file (here, we call it myfun.m again):

  ```
  function z=myfun(x,y)
  z=x.^2.*y;
  ```

  Then call it in Matlab using `@`

  ```
  Q=dblquad(@myfun,0,3,1,2);
  ```

The double quadrature formula only works on rectangular regions in the plane. How would we integrate over non-square regions? It depends on the complexity of the region. Here are two options- The first is where the region is somewhat simple (a circle in the plane), and the second is more general.

1. The volume of a hemisphere: $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-(x^2+y^2)}\,dy\,dx$

   In this case, the integrand is somewhat simple, and we can extend the definition of the integrand so that it is zero outside the area of interest. Mathematically speaking,

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-(x^2+y^2)}\,dy\,dx = \int_{-1}^{1}\int_{-1}^{1} G(x,y)\,dy\,dx$$

   where

$$G(x,y) = \begin{cases} \sqrt{1-(x^2+y^2)} & \text{if } x^2+y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   In Matlab, the expression `(x.^2+y.^2<=1)` is a logical statement- It returns "1" if the expression is TRUE, "0" if FALSE. Therefore, to find the volume, we could type:

```
f3=@(x,y)sqrt(1-(x.^2+y.^2)).*(x.^2+y.^2<=1);
Q=dblquad(f3,-1,1,-1,1);
```

2. Alternatively (or more generally), given:

$$\int_a^b \int_{y=g_1(x)}^{y=g_2(x)} F(x,y)\,dy\,dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} F(x,y)\,dy \right) dx = \int_a^b G(x)\,dx$$

   That is, given a numerical value of x, we would write the function $G(x)$ as:

$$G(x) = \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy$$

   Here's a specific example: $\int_0^2 \int_{x^2}^{2x} x^2+y^2\,dy\,dx$

   We write the M-file for the inside integral:

```
function z=myfun(x)
n=length(x);
z=zeros(size(x));
for j=1:n
    a=x(j)^2;
    b=2*x(j);
    h=@(y)(a+y.^2);
    z(j)=quad(h,a,b);
end
```

In Matlab, type: `Q=quad(@myfun,-1,1);` and you may get some warnings, but we do get an answer (the exact value is 216/33).

3. Here's another example of this second type, where we build a function to evaluate the inside integral:

   > Evaluate the volume of the solid under the surface $z = x^3y^4 + xy^2$ and above the region bounded by the curves $y = x^3 - x$ and $y = x^2 + x$ for $x \geq 0$.

   Doing our normal calculus thing, we see that the region of integration is for x ranging from 0 to 2, the parabola is the top function and the cubic is the bottom function:

   $$\int_0^2 \int_{x^3-x}^{x^2+x} x^3y^4 + xy^2 \, dy \, dx \approx 961.1809$$

   Now write the M-file for the integrand. Think of the expressions in x as representing parameters:

   ```
   function z=myfun2(x)
   n=length(x);
   z=zeros(size(x));
   for j=1:n
       a=x(j)^3-x(j);
       b=x(j)^2+x(j);
       c=x(j)^3;
       d=x(j);
       h=@(y)(c*y.^4+d*y.^2);
       z(j)=quad(h,a,b);
   end
   ```

   In the command window, type `Q=quad(@myfun2,0,2);`

Example: Triple Integral

Here is an example problem: Evaluate

$$\int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2} \, dz \, dy \, dx$$

(This is a sideways paraboloid (y axis running through the "top"), cut off at $y = 4$. For future reference, this could be evaluated exactly using polar coordinates:

$$\int_0^{2\pi} \int_0^2 (4 - r^2) r\, r\, dr\, d\theta = \frac{128\pi}{15}$$

Using the built-in triple integral

To call Matlab's `triplequad`, your function must be able to handle a vector x input, and scalars y, z. In this example, we have the following:

```
function A=tripletrialfunc(x,y,z)
%Assume vector x and scalars y and z
%

%See if (x,y,z) is in the correct region.  If not, return 0.
N=length(x);
for j=1:N
  flag=0;
    if x(j)<=2 & x(j)>=-2
      if y>=x(j)^2 & y<=4
          if abs(z)<=sqrt(y-x(j)^2)
              flag=1;
          end
      end
    end
  if flag==0
      A(j)=0;
  else
      A(j)=sqrt(x(j).^2+z.^2);
  end
end
```

Output:

```
>> tic;
A=triplequad(@tripletrialfunc,-2,2,-4,4,-4,4);
toc

Elapsed time is 38.867135 seconds.
```

This took a LONG time to run- Probably due to the triple loop and the requirement to check to see if (x, y, z) is in the region of integration.

As an alternative, we can simply call `quad` three times. Notice that, with a fixed value of $x = a$, the inside two integrals become:

$$\int_{a^2}^{4} \int_{-\sqrt{y-a^2}}^{\sqrt{y-a^2}} \sqrt{a^2 + z^2}\, dz\, dy$$

Furthermore, if y was a fixed parameter, b, then the inside integral would simply be:

$$\int_{-\sqrt{b-a^2}}^{\sqrt{b-a^2}} \sqrt{a^2 + z^2}\, dz$$

This is implemented by the pair of functions below, saved as `middle.m`:

```
function B=middle(x)
%Evaluates the middle integral, then calls the inside function
N=length(x);
for j=1:N
    y1=x(j).^2;
    y2=4;
    a=x(j);
    h=@(y) Inside(y,a);
    B(j)=quad(h,y1,y2);
end

function A=Inside(y,c)
%  Evaluates the inside integral
  c=c.^2; %Coming from the outside parameter (this is fixed x)
  N=length(y);
  for j=1:N
    b=sqrt(y(j)-c);
    a=-b;
    h=@(z)sqrt(c+z.^2);
    A(j)=quad(h,a,b);
  end
```

In the Matlab command window, we integrate the middle integral with respect to x (and the rest of the integration is done by our two functions). HINT: First type

```
warning off
```

because of the “singularities” in our function.

```
>> tic; A=quad(@middle,-2,2); toc
Elapsed time is 4.315762 seconds.
```

By comparison, here are the numerical values (Matlab’s default tolerance is set to 10^{-6}):

```
>> Exact=128*pi/15
   26.8083
>> Exact-A1   %Using triplequad
 -5.3277e-005
>> Exact-A2   %Using the custom three quads
  1.8277e-005
```