Questions to Consider- Exam 1

1. How many multiplications and additions/subtractions are there if we evaluate the polynomial:
$$P(x) = 3x^4 + 4x^3 + 5x^2 - 5x + 1$$
without nesting (direct evaluation)? How many flops by using Horner's Method?

2. Prove the *attracting fixed point theorem*: If a function g satisfies the following
    - is differentiable over the reals,
    - has a fixed point r
    - has an interval I about r so that
    $$|g'(x)| \leq \lambda < 1$$
    for all x in I,

    then, for $x_0 \in I$, function iteration $x_{n+1} = g(x_n)$ produces a sequence that converges to r.

    (NOTE: This is a special case of the more general theorem proven in class. You may assume the Value theorems).

3. Using your previous proof, show that in general, fixed point iteration converges linearly.

4. Using a proof similar to Question 1, show that $g'(r) = 0$ implies quadratic convergence (HINT: Use a second order Taylor expansion of g. That is, $g(x) = g(r) + \cdots$)

5. Discuss how the attracting fixed point theorem applies to Newton's method. In particular, discuss the two cases: (i) r is a simple root of f, and (ii) r is a root of multiplicity m for f.

6. How is the IVT and MVT used together to prove the existence of some exact number of roots? For example, how would you prove that $f(x) = 1 - x^2$ has exactly two roots between $x = -2$ and $x = 2$ (assuming of course that we don't know what they are!).

7. Let $f(x) = \cos(x)$. If we approximate the root by $x_c = 1.56$, find the forward error, the backward error.

8. Explain the Bisection method. It is very useful because of the error approximation (what is it?). What kind of convergence do we get (and show it):

9. Give the Taylor polynomial of degree 3 with remainder (in general):

10. Suppose $f(-1)f(2) < 0$ and f is continuous on the reals. Using Bisection, how many iterations are necessary to guarantee that our approximation is correct within 7 decimal places?

11. Find the c in that theorem, if $f(x) = e^{x/2}$, the polynomial is based at $x = 0$, and we want to approximate $f(0.2)$.

12. (Exercise 9, p. 23)

13. Let $x_n = \frac{1}{n^2}$. Give the rate of convergence of x_n to 0.

14. Give an example of a sequence that converges to zero of order 3 (and show it):

15. Why is the Wilkinson polynomial a famous example in numerical analysis? (What does it illustrate?)

16. Using Newton's Method, if x_0 begins close to $r = 1$ for $f(x) = x^2 - 1$, what are the limits:
$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|} \qquad \lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^2} \qquad \lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^3}$$

17. Using Newton's Method, if x_0 begins close to $r = 3$ for $f(x) = (x+2)(x-3)^2$, what is
$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|} \qquad \lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^2} \qquad \lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^3}$$

18. We said in class that the number of flops to perform Gaussian elimination is approximately $\frac{1}{3}n^3$. Given that, how much more does it take to eliminate a system if the number of equations (and variables) is doubled?

19. Suppose that the Taylor (actually Maclaurin) series for a function is:
$$f(x) = x + \frac{1}{2}x^3 + \frac{1}{3}x^5 + \frac{1}{4}x^7 + \cdots$$
and
$$g(x) = 1 - x^2 + x^4 - x^6 + \cdots$$
Consider $F(x) = f(x) - xg(x)$. What is the multiplicity of the root $x = 0$?

20. Assume that for the secant method the following is true for some $C > 0$:
$$e_n \approx Ce_{n-1}$$
Show that this implies that the order of convergence is $\alpha = \frac{1+\sqrt{5}}{2}$

21. Suppose we come up with a new algorithm to compute the value of a function at a given point, $y = f(x)$. Many times of course, we will not have an exact computation, so we can expect an error. In this "problem", what should the forward error be? What should the backward error be? (HINT: Not the same as before).

22. Section 2.2 Exercises (p. 90), 1(a), 2(a).