

Exam 1 Review SOLUTIONS

1. How many multiplications and additions/subtractions are there if we evaluate the polynomial:

$$P(x) = 3x^4 + 4x^3 + 5x^2 - 5x + 1$$

without nesting (direct evaluation)? How many flops by using Horner's Method?

SOLUTION: Without nesting, we have 10 multiplications and 4 add/subtracts, which gives a total of 14.

With nesting, we evaluate:

$$1 - 5(x + 5(x + 4(x + 3x)))$$

and now we have 8 flops (4 multiplications, 4 add/subtracts).

2. Prove the *attracting fixed point theorem*: If a function g satisfies the following

- is differentiable over the reals,
- has a fixed point r
- has an interval I about r so that

$$|g'(x)| \leq \lambda < 1$$

for all x in I ,

then, for $x_0 \in I$, function iteration $x_{n+1} = g(x_n)$ produces a sequence that converges to r .

(NOTE: This is a special case of the more general theorem proven in class. You may assume the Value theorems).

SOLUTION:

First of all, if the sequence $x_{n+1} = g(x_n)$ converges at all, it must converge to a fixed point of g , since convergence would imply that $g(r) = r$.

We show convergence by showing that the distance between x_n and the fixed point r goes to zero as $n \rightarrow \infty$. Since g is assumed to be differentiable on the reals, it satisfies the hypotheses of the Mean Value Theorem, and so for any x , there is a c between x and r so that

$$g(x) - g(r) = g'(c)(x - r)$$

In particular, if $x_0 \in I$, then:

$$g(x_0) - g(r) = g'(c_0)(x_0 - r) \quad \Rightarrow \quad |x_1 - r| = |g'(c_0)| |x_0 - r| \leq \lambda |x_0 - r|$$

Similarly, since the distance between x_1 and r is smaller than the distance from x_0 to r , then x_1 is still in I , and

$$|x_2 - r| = |g'(c_1)| |x_1 - r| \leq \lambda^2 |x_0 - r|$$

And we repeat this process to see that at step n ,

$$|x_n - r| \leq \lambda^n |x_0 - r|$$

Since $\lambda < 1$, $|x_n - r| \rightarrow 0$ as $n \rightarrow \infty$.

3. Using your previous proof, show that in general, fixed point iteration converges linearly.

SOLUTION: The idea is to use our first line, which uses the Mean Value Theorem:

$$|g(x_n) - g(r)| = |g'(c_n)| |x_n - r| \Rightarrow \frac{|x_{n+1} - r|}{|x_n - r|} = |g'(c_n)|$$

for some c_n between x_n and r . Take the limit, and assume that $g'(x)$ is continuous at r :

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - r|}{|x_n - r|} = \lim_{n \rightarrow \infty} |g'(c_n)| = |g'(r)|$$

Now, if $g'(r) \neq 0$, we have linear convergence of x_n to r . Recall that in the definition of the order of convergence, we had to make sure that

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} \neq 0$$

4. Using a proof similar to Question 1, show that $g'(r) = 0$ implies quadratic convergence (HINT: Use a second order Taylor expansion of g . That is, $g(x) = g(r) + \dots$)

SOLUTION: We start with a line like Question 1, and by Taylor's Theorem with remainder, there is a c between x and r so that

$$g(x) = g(r) + g'(r)(x - r) + \frac{g''(c)}{2}(x - r)^2$$

Simplify and rearrange terms, and take advantage of $g'(r) = 0$:

$$\frac{g(x) - g(r)}{(x - r)^2} = \frac{g''(c)}{2}$$

Now substitute x_n for x , and x_{n+1} for $g(x_n)$, and use the fact that $r = g(r)$. For each n , we have a c_n between x_n and r such that:

$$\frac{|x_{n+1} - r|}{|x_n - r|^2} = \frac{g''(c_n)}{2}$$

Take the limit of both sides, and assume g'' is continuous at r :

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - r|}{|x_n - r|^2} = \lim_{n \rightarrow \infty} \frac{|g''(c_n)|}{2} = \frac{|g''(r)|}{2}$$

5. Discuss how the attracting fixed point theorem applies to Newton's method. In particular, discuss the two cases: (i) r is a simple root of f , and (ii) r is a root of multiplicity m for f .

SOLUTION: Newton's method is a special case of fixed point iteration. If we are using Newton's method to find the root of a function f , then the Newton iteration is defined by:

$$x_{n+1} = N(x_n) \quad \text{where} \quad N(x) = x - \frac{f(x)}{f'(x)}$$

We should establish some facts:

- The fixed point of N corresponds to the root of f . If r is a simple root,

$$r = r - \frac{f(r)}{f'(r)} \Leftrightarrow f(r) = 0$$

- $N'(r) = 0$, so by the attracting fixed point theorem, we get quadratic convergence to the root of f .

$$N'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}$$

$$\text{so } N'(r) = 0.$$

If r is a root of multiplicity m , then $N'(r) = \frac{m-1}{m}$ and therefore the convergence is only linear.

6. How is the IVT and MVT used together to prove the existence of some exact number of roots? For example, how would you prove that $f(x) = 1 - x^2$ has exactly two roots between $x = -2$ and $x = 2$ (assuming of course that we don't know what they are!).

SOLUTION: As in the homework, we use the IVT to establish that there must be at least 1 root, and the MVT to establish that there can be at most one root (in a given interval). In this example,

$$f(-2) = -3 < 0 \quad \text{and} \quad f(0) = 1 > 0$$

Since $f(x) = 1 - x^2$ is continuous everywhere, the IVT applies and we conclude there is a root of f in the interval $(-2, 0)$.

Similarly,

$$f(0) = 1 > 0 \quad \text{and} \quad f(2) = -3 < 0$$

and we again conclude there is at least one root of f , this time in the interval $(0, 2)$.

Without loss of generality, we will prove that there is exactly one root in $(0, 2)$, and the same argument applies to the other interval. The derivative is $f'(x) = -2x$, therefore $f'(x) < 0$ for all x in $(0, 2)$. Suppose there were two (or more) roots, r_1 and r_2 in $(0, 2)$.

Then, since f is differentiable for all real numbers, the Mean Value Theorem applies, and there must be a c in $(0, 2)$ such that

$$0 = \frac{f(r_2) - f(r_1)}{r_2 - r_1} = f'(c)$$

But there is no such c in $(0, 2)$. Therefore, f has at most one root in $(0, 2)$.

Since there is both at most one root and at least one root in $(0, 2)$, there must be exactly one root in $(0, 2)$.

7. Let $f(x) = \cos(x)$. If we approximate the root by $x_c = 1.56$, find the forward error, the backward error.

SOLUTION: The forward error for the root finding problem is $|x_c - r|$, and the backward error is $|\cos(x_c)|$. In this case, we use r that is closest to x_c , at $\pi/2$. Therefore, the backward error is approximately 0.0107961 and the forward error is approximately 0.0107963 (NOTE: Because the two errors were so close, I included more digits than one would normally include).

8. Explain the Bisection method. It is very useful because of the error approximation (what is it?). What kind of convergence do we get (and show it):

SOLUTION: The bisection method is used to solve for the root of a function $f(x)$ on an interval (a, b) , where we know that the signs of $f(a)$ and $f(b)$ are different (this guarantees the existence of a root in the interval as long as f is continuous there). We get a better approximation by determining which half of the interval still contains the interval, and iterate. That is, if $c = (a + b)/2$, the new interval is either (a, c) or (c, b) depending on which pair still contains the root.

The error approximation is:

$$|x_n - r| \leq \frac{|b - a|}{2^{n+1}}$$

This is very nice- We don't have this kind of a maximum error for the other algorithms in Chapter 1.

9. Give the Taylor polynomial of degree 3 with remainder (in general):

SOLUTION:

For the function $f(x)$ based at $x = a$,

$$f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \frac{f^{(iv)}(c)}{4!}(x - a)^4$$

where c is between x and a .

10. Suppose $f(-1)f(2) < 0$ and f is continuous on the reals. Using Bisection, how many iterations are necessary to guarantee that our approximation is correct within 7 decimal places?

SOLUTION:

Generally speaking, for n iterations on (a, b) and p decimal places, we would have:

$$\frac{b-a}{2^{n+1}} < \frac{1}{2} \times 10^{-p}$$

Substitute in the numbers and solve for n :

$$\frac{2 - (-1)}{2^{n+1}} < \frac{10^{-7}}{2} \Rightarrow n > 24.84$$

From this, we see that we need 25 iterations to meet our error criteria.

11. Find the c in that theorem, if $f(x) = e^{x/2}$, the polynomial is based at $x = 0$, and we want to approximate $f(0.2)$.

Typo: This problem was meant to be a continuation of Question 9, but Question 10 got in the way.

The Taylor expansion with error term is:

$$e^{x/2} = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{e^{c/2}}{4! \cdot 16}x^4$$

Substitute $x = 0.2$ and solve for c :

$$1.10516 + .416 \times 10^{-5}e^{c/2} = 1.1051709$$

You should get that c is approximately 0.04 (which is between 0 and 0.2).

12. (Exercise 9, p. 23)

SOLUTION:

This one is similar to the previous one. Go ahead and assume that our interval is between 0 and 0.02 (that is where our estimate is). In this case, the Taylor remainder is

$$E = \frac{x^2}{8(1+c)^{3/2}}$$

for c between 0 and 0.02. Setting $x = 0.02$, we get a maximum error when the denominator is smallest, $c = 0$. Therefore, the upper bound is:

$$E = \frac{(0.02)^2}{8(1)^{3/2}} = 0.00005$$

The actual values are $\sqrt{1.02} \approx 1.0099505$ and $1 + \frac{1}{2}(0.02) = 1.01$, which is a difference of 0.0000495, slightly less than the upper bound E .

13. Let $x_n = \frac{1}{n^2}$. Give the rate of convergence of x_n to 0.

SOLUTION:

If $p_n = \frac{1}{n^2}$, then

$$\frac{|p_{n+1}|}{|p_n|^\alpha} = \frac{n^{2\alpha}}{(n+1)^2} = \left(\frac{n^\alpha}{n+1}\right)^2$$

This converges to 0 if $\alpha < 1$, converges to 1 if $\alpha = 1$, and diverges if $\alpha > 1$. Therefore, we have linear convergence ($\alpha = 1$).

14. Give an example of a sequence that converges to zero of order 3 (and show it):

SOLUTION: One possibility: 2^{-3^n} converges cubically to 0.

$$\frac{|p_{n+1}|}{|p_n|^\alpha} = \frac{2^{-3^{n+1}}}{2^{\alpha \cdot 3^n}} = 2^{\alpha \cdot 3^n - 3^{n+1}} = 2^{3^n(\alpha - 3)}$$

If $\alpha < 3$, the exponent is negative and the ratio converges to zero. If $\alpha = 3$, the exponent is zero and the ratio converges to 1. If $\alpha > 3$, the ratio diverges. Therefore, we have cubic convergence.

15. Why is the Wilkinson polynomial a famous example in numerical analysis? (What does it illustrate?)

SOLUTION: The famous quote is that Wilkinson regarded running into this polynomial as the worst experience in his career. The roots are very simple (some positive integers), but there is huge error when trying to compute them. (I will not ask you to use the Wilkinson polynomial, this question was to give you an example other than Figure 1.7 on pg. 47).

16. Using Newton's Method, if x_0 begins close to $r = 1$ for $f(x) = x^2 - 1$, what are the limits:

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - r|}{|x_n - r|} \quad \lim_{n \rightarrow \infty} \frac{|x_{n+1} - r|}{|x_n - r|^2} \quad \lim_{n \rightarrow \infty} \frac{|x_{n+1} - r|}{|x_n - r|^3}$$

SOLUTION:

Simple root- Newton's method converges quadratically. Therefore, the first limit is zero, the second is either $|N''(1)|/2$ or $|f''(1)|/|2f'(1)|$ which is easier to compute: $1/2$. The third limit is ∞ .

17. Using Newton's Method, if x_0 begins close to $r = 3$ for $f(x) = (x+2)(x-3)^2$, what is

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - r|}{|x_n - r|} \quad \lim_{n \rightarrow \infty} \frac{|x_{n+1} - r|}{|x_n - r|^2} \quad \lim_{n \rightarrow \infty} \frac{|x_{n+1} - r|}{|x_n - r|^3}$$

SOLUTION:

Multiple root, so $N'(3) = \frac{2-1}{2} = \frac{1}{2}$, and we have linear convergence to $N'(3) = 1/2$

18. We said in class that the number of flops to perform Gaussian elimination is approximately $\frac{1}{3}n^3$. Given that, how much more does it take to eliminate a system if the number of equations (and variables) is doubled?

SOLUTION:

If n is changed to $2n$, then the number of flops changes from $(1/3)n^3$ to:

$$\frac{1}{3}(2n)^3 = 8 \cdot \frac{1}{3}n^3$$

so it takes 8 times as many flops.

19. Suppose that the Taylor (actually Maclaurin) series for a function is:

$$f(x) = x + \frac{1}{2}x^3 + \frac{1}{3}x^5 + \frac{1}{4}x^7 + \cdots$$

and

$$g(x) = 1 - x^2 + x^4 - x^6 + \cdots$$

Consider $F(x) = f(x) - xg(x)$. What is the multiplicity of the root $x = 0$?

SOLUTION:

Recall that a root r has multiplicity m if $f(r) = 0$, $f'(r) = 0, \dots, f^{(m-1)}(r) = 0$, but $f^{(m)}(r) \neq 0$. Also, in that case, we can factor f as:

$$f(x) = (x - r)^m G(x)$$

where $G(r) \neq 0$. In this case,

$$F(x) = x^3 \left(\frac{3}{2} - \frac{2}{3}x^2 + \cdots \right) = x^3 G(x)$$

Therefore, F has a root at zero of multiplicity 3.

20. Assume that for the secant method the following is true for some $C > 0$:

$$e_n \approx C e_{n-1}$$

Error (Corrected in Class Notes) This should be:

$$e_{n+1} = C e_n e_{n-1}$$

Then we want to show that

$$e_{n+1} = \kappa e_n^\alpha$$

and find α . This takes a little manipulation/substitution:

$$e_{n+1} = \kappa e_n^\alpha \Rightarrow e_n = \kappa e_{n-1}^\alpha$$

Therefore,

$$ke_n^\alpha = e_{n+1} = Ce_n e_{n-1} = \hat{c}e_{n-1}^\alpha e_{n-1} \Rightarrow e_n = \tilde{c}e_{n-1}^{(\alpha+1)/\alpha}$$

Now, e_n was also equal to ke_{n-1}^α , so that means (equating exponents):

$$\frac{\alpha+1}{\alpha} = \alpha \Rightarrow \alpha+1 = \alpha^2$$

From which we get

$$\alpha = \frac{1+\sqrt{5}}{2}$$

21. Suppose we come up with a new algorithm to compute the value of a function at a given point, $y = f(x)$. Many times of course, we will not have an exact computation, so we can expect an error. In this “problem”, what should the forward error be? What should the backward error be? (HINT: Not the same as before).

SOLUTION: In the problem of root finding, we said:

$$\text{Given } f(x) = 0 \rightarrow \text{ALGORITHM} \rightarrow \text{Solution: } x_c \approx r$$

In this case, the forward error was $|x_c - r|$ and the backward error was $|f(x_c)|$.

In the new situation, we have:

$$\text{Given } y = f(x) \rightarrow \text{ALGORITHM} \rightarrow \text{Solution: } y_c \approx y$$

In this case, the forward error is $|y_c - y|$ (change in y), and the backward error is the corresponding change in the domain (change in x)- That is, we think of $y = f(x)$ and $y_c = f(x + \Delta x)$ for some Δx (more on this later- This question is here to get you to recall how we defined forward and backward error).

22. Section 2.2 Exercises (p. 90), 1(a), 2(a).

(Use Maple to check your answers).