Notes on Exam 2

There will be two parts: One part of the exam will be a take home exam. It will be posted on our class website sometime on Wednesday. You may use the text, class notes, and Matlab, but you should refrain from using the internet or classmates- Your work should be your own. Solutions will be due the following Monday at the beginning of class.

The in-class portion of the exam will be 50 minutes. There will be some "proof", some computation. You should bring a calculator with you (anything that will do basic arithmetic), and you may bring a 3 inch by 5 inch card (both sides) with any notes you wish to make (alternatively, you may use one sheet with no more than 30 square inches of written material, like 6×5).

Be sure you look over the assigned homework!

Exam 2 will cover the material from Sections:

- Section 2.3: Vector norms, matrix norms, backward/forward error, error magnification factor, condition number (also see the handouts from class).
- Section 2.4 $PA = LU$ will only be covered to the extent that the matrix P implements row swaps- Those were missing in our regular LU decomposition from the earlier material.
- Solving $A\mathbf{x} = \mathbf{b}$ using iterative methods:
  - Jacobi
  - Gauss-Seidel
  - General background theory for iterating (handout)
- Sparse matrices (Matlab Only)
- Solve (nonlinear) systems: Multidimensional Newton's method.
- Interpolation:
  - Method: Lagrange Polynomials
  - Method: Newton's Divided Differences
  - Error and error bounds
  - The Runge Phenomenon
  - Chebyshev Points (not polynomials)
  - Cubic splines (know the conditions, everything else will be associated with Matlab).

Some Review Questions

These are here to just help with the kinds of questions you might encounter during the exam. They should not be taken to be completely exhaustive.

1. Show that $|det(A)|$ does not satisfy the requirements to be a norm for a matrix A.

2. Rearrange the equations to form a strictly diagonally dominant system. Apply two steps of Jacobi and Gauss-Seidel methods starting with the zero vector:

$$\begin{aligned} u + 3v &= -1 \\ 5u + 4v &= 6 \end{aligned}$$

3. For the matrix $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 0 & -1 \\ 2 & 1 & -2 \end{bmatrix}$, find a vector $\mathbf{x}$ so that:

   (a) $\|A\|_\infty = \|A\mathbf{x}\|_\infty / \|\mathbf{x}\|_\infty$

   (b) $\|A\|_1 = \|A\mathbf{x}\|_1 / \|\mathbf{x}\|_1$

4. Use your answer from Exercise 2 (from the Jacobi method) and compute the relative forward, relative backward, and error magnification factor.

5. Suppose we iterate: $\mathbf{x}_{n+1} = T\mathbf{x}_n + \mathbf{c}$. If it converges, what will it converge to?

6. Give a justification for each of the appropriate lines below, which together will constitute a proof of the following:

   Theorem: If $\rho(T) < 1$, then $(I - T)^{-1}$ exists, and the sequence $\mathbf{x}_{n+1} = T\mathbf{x}_n + c$ converges to the fixed point.

   Proof:

   - If λ is an eigenvalue of T, then $1 - \lambda$ is an eigenvalue of $I - T$ (Give justification:)
   - If $I - T$ is not invertible, then $\lambda = 1$ is an eigenvalue of I_T (Give justification:)
   - All of the eigenvalues of T are less than 1 (in abs value) (Give justification:)
   - If we assume that T is convergent (Give the definition) if $rho(T) < 1$, then show that

   $$(I - T)^{-1} = I + T + T^2 + \ldots$$

   Hint: Think about the proof for the formula of a Geometric series,

   $$\frac{1}{1 - r} = 1 + r + r^2 + r^2 + \ldots$$

- Show that:
$$\mathbf{x}_n = T^n\mathbf{x}_0 + \left(T^{n-1} + T^{n-2} + \cdots + T^2 + T + I\right)\mathbf{c}$$
and take the limit of both sides to see that the system converges:
$$\lim_{n\to\infty} \mathbf{x}_n = \lim_{n\to\infty} T^n\mathbf{x}_0 + \lim_{n\to\infty}\left(T^{n-1} + T^{n-2} + \cdots + T^2 + T + I\right)\mathbf{c} =$$
$$\mathbf{0} + (I-T)^{-1}\mathbf{c}$$

7. Give a sketch of a proof of the following, which is a generalization of a well known theorem in Calc I:

   Let $f, f', \ldots, f^{(n)}$ all be continuous on $[a,b]$, and suppose we have $n+1$ roots of f in $[a,b]$. Then there exists $c \in (a,b)$ so that $f^{(n)}(c) = 0$.

8. Let $x_0, x_1, \ldots, x_n$ be $n+1$ real numbers, and let x be fixed (so its also constant). Consider the function of t:
$$\Psi(t) = \frac{(t-x_0)(t-x_1)\cdots(t-x_n)}{(x-x_0)(x-x_1)\cdots(x-x_n)}$$
   (a) $\Psi(t)$ is a polynomial of what degree?
   (b) What is $\Psi(x_j)$, $j = 0, 1, 2, \ldots, n$? What is $\Psi(x)$?
   (c) What is the $n+1$st derivative of Ψ, with respect to t?

9. Prove the error formula for polynomial interpolation (you should use the previous exercises!):

   Given $n+1$ points,
$$(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$$
   and suppose that $P(x)$ is the (n^{th} degree) polynomial interpolating those points. Use the following:
$$g(t) = f(t) - P(t) - (f(x) - P(x))\Psi(t) \tag{1}$$
   to show that there is a c so that
$$f(x) = P(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)\cdots(x-x_n)$$
   Hint: Consider the $n+1$st derivative of g.

10. Prove the "Main Theorem of Polynomial Interpolation": Given $(x_1, y_1), \ldots, (x_n, y_n)$ with distinct values of x, there is one and only one polynomial P of degree at most $n-1$ that interpolates the data.

    SET UP: We already know such a polynomial exists by the Lagrange formula (exists for distinct values of x), so we only need to prove the uniqueness. Start by assuming that there are two interpolating polynomials of degree $n-1$ or less, $P(x)$ and $Q(x)$ and consider the polynomial: $P(x) - Q(x)$. How many zeros does $P - Q$ have?

11. (Exercise 1, p. 181) Decide whether the following equations form a cubic spline:

$$S(x) = \begin{cases} x^3 + x - 1 & \text{if } 0 \le x \le 1 \\ -(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1 & \text{if } 1 < x \le 2 \end{cases}$$

12. What is the "not-a-knot" condition for the cubic spline?

13. Find the linearization of F at $(u, v) = (1, 1)$:

$$F(u, v) = (u + \mathrm{e}^{u-v}, 2u + v)$$

14. Perform one step of Newton's Method on the nonlinear system:

$$\begin{aligned} u^3 - v^3 + u &= 0 \\ u^2 + v^2 &= 1 \end{aligned}$$

    beginning with the vector $(u, v) = (1, 1)$

15. Give an example, or explain why it doesn't exist:

    (a) A polynomial of degree 6 that is zero at $x = 1, 2, 3, 4, 5, 6$ and is 10 at $x = 7$.

    (b) A polynomial of degree 6 that is zero at $x = 1, 2, 3, 4, 5, 6$ and is 10 at $x = 7$, and is 70 at $x = 8$.

16. Write the 7 Chebsyhev points if we are to interpolate a function on the interval $[-1, 5]$.

17. What is the worst error one can expect by using a polynomial of degree 5 to approximate the sine function on the interval $[0, 2]$ using the Chebyshev points?