

SOLUTIONS to Review Set 2, Math 467

These are here to just help with the kinds of questions you might encounter during the exam. They should not be taken to be completely exhaustive. Be sure you look over your homework!

1. Show that $|\det(A)|$ does not satisfy the requirements to be a norm for a matrix A .

SOLUTION:

The norm is always non-negative? Yes. It is zero only for the zero matrix? No (in fact, that's as far as we need to take it).

2. Rearrange the equations to form a strictly diagonally dominant system. Apply two steps of Jacobi and Gauss-Seidel methods starting with the zero vector:

$$\begin{aligned}u + 3v &= -1 \\ 5u + 4v &= 6\end{aligned}$$

SOLUTIONS: To be strictly diagonally dominant, swap equations first. We then proceed to Jacobi iteration:

$$\begin{aligned}5u + 4v &= 6 \\ u + 3v &= -1\end{aligned} \quad \Rightarrow \quad \begin{aligned}u &= \frac{1}{5}(6 - 4v) \\ v &= \frac{1}{3}(-1 - u)\end{aligned}$$

Now iterate to get

$$u_1 = \frac{6}{5}, \quad v_1 = -\frac{1}{3}, \quad u_2 = \frac{22}{15}, \quad v_2 = \frac{-11}{15}$$

3. For the matrix $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 0 & -1 \\ 2 & 1 & -2 \end{bmatrix}$, find a vector \mathbf{x} so that:

(a) $\|A\|_\infty = \|A\mathbf{x}\|_\infty / \|\mathbf{x}\|_\infty$

SOLUTION: If $\mathbf{x} = [-1, 1, 1]^T$, then

$$\|A\mathbf{x}\|_\infty = \max\{6, 0, -3\} = 6$$

which is also the matrix infinity norm.

(b) $\|A\|_1 = \|A\mathbf{x}\|_1 / \|\mathbf{x}\|_1$

SOLUTION: Recall that

$$A\mathbf{x} = x_1 \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

To make the matrix norm equal to the largest column norm, choose $\mathbf{x} = [0, 0, 1]^T$.

4. Use your answer from Exercise 2 (from the Jacobi method) and compute the relative forward, relative backward, and error magnification factor.

SOLUTION: The true solution to the system is $\mathbf{x} = [2, -1]^T$. The norm is not specified, so as in the text we'll use the infinity norm.

$$\frac{\|x^c - x\|}{\|x\|} = \frac{4/5}{2} = \frac{2}{5}$$

$$\frac{\|Ax^c - b\|}{\|b\|} = \frac{8/3}{6} = \frac{4}{9}$$

$$\text{EMF} = \frac{2/5}{4/9} = \frac{9}{10}$$

And to compute the condition number, compute the inverse of A :

$$A^{-1} = \frac{1}{11} \begin{bmatrix} 3 & -4 \\ -1 & 5 \end{bmatrix}$$

so that

$$\text{cond}(A) = \|A\| \|A^{-1}\| = 9 \cdot \frac{7}{11} = \frac{63}{11}$$

5. Suppose we iterate: $\mathbf{x}_{n+1} = T\mathbf{x}_n + \mathbf{c}$. If it converges, what will it converge to?

SOLUTION: If it converges, it will converge to its fixed point:

$$x = Tx + c \Rightarrow (I - T)x = c \Rightarrow x = (I - T)^{-1}c$$

6. Give a justification for each of the appropriate lines below, which together will constitute a proof of the following:

Theorem: If $\rho(T) < 1$, then $(I - T)^{-1}$ exists, and the sequence $\mathbf{x}_{n+1} = T\mathbf{x}_n + c$ converges to the fixed point.

Proof:

- If λ is an eigenvalue of T , then $1 - \lambda$ is an eigenvalue of $I - T$ (Give justification:)

SOLUTION:

$$T\mathbf{x} = \lambda\mathbf{x} \Leftrightarrow \mathbf{x} - T\mathbf{x} = \mathbf{x} - \lambda\mathbf{x} \Leftrightarrow (I - T)\mathbf{x} = (1 - \lambda)\mathbf{x}$$

- If $I - T$ is not invertible, then $\lambda = 1$ is an eigenvalue of $I - T$ (Give justification:)

SOLUTION:

If $I - T$ is not invertible, then one of its eigenvalues are 0: $\mu = 0$. Now, $T = I - (I - T)$, so $\lambda = (1 - \mu)$ is an eigenvalue of T , which would be 1 if $\mu = 0$.

- All of the eigenvalues of T are less than 1 (in abs value) (Give justification:)

SOLUTION:

We assumed that $\rho(T) < 1$, and $\rho(T)$ is the spectral radius of T . The spectral radius is the magnitude of the largest eigenvalue, so if it is less than one, all eigenvalues are less than one (in magnitude).

- If we assume that T is convergent (Give the definition) if $\rho(T) < 1$, then show that

$$(I - T)^{-1} = I + T + T^2 + \dots$$

Hint: Think about the proof for the formula of a Geometric series,

$$\frac{1}{1 - r} = 1 + r + r^2 + r^3 + \dots$$

SOLUTION:

$$(I - T)(I + T + T^2 + \dots + T^n) = I - T^{n+1}$$

Take the limit of both sides:

$$(I - T) \lim_{n \rightarrow \infty} (I + T + T^2 + \dots + T^n) = \lim_{n \rightarrow \infty} I - T^{n+1} = I$$

Therefore

$$(I - T)^{-1} = \lim_{n \rightarrow \infty} (I + T + T^2 + \dots + T^n)$$

- Show that:

$$\mathbf{x}_n = T^n \mathbf{x}_0 + (T^{n-1} + T^{n-2} + \cdots + T^2 + T + I) \mathbf{c}$$

and take the limit of both sides to see that the system converges:

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{x}_n &= \lim_{n \rightarrow \infty} T^n \mathbf{x}_0 + \lim_{n \rightarrow \infty} (T^{n-1} + T^{n-2} + \cdots + T^2 + T + I) \mathbf{c} = \\ &\mathbf{0} + (I - T)^{-1} \mathbf{c} \end{aligned}$$

SOLUTION:

To show the first item, list the first few points of the orbit:

$$\begin{aligned} \mathbf{x}_1 &= T\mathbf{x}_0 + \mathbf{c} \\ \mathbf{x}_2 &= T\mathbf{x}_1 + \mathbf{c} = T(T\mathbf{x}_0 + \mathbf{c}) + \mathbf{c} = T^2\mathbf{x}_0 + (T + I)\mathbf{c} \\ \mathbf{x}_3 &= T\mathbf{x}_2 + \mathbf{c} = T(T^2\mathbf{x}_0 + (T + I)\mathbf{c}) + \mathbf{c} = T^3\mathbf{x}_0 + (T^2 + T + I)\mathbf{c} \end{aligned}$$

Continuing in this manner, we get the desired result:

$$\mathbf{x}_n = T^n \mathbf{x}_0 + (T^{n-1} + \cdots + T^2 + T + I)\mathbf{c}$$

In taking the limit, we would have to know that T is a convergent matrix, and then we would get the conclusion above, with

$$\lim_{n \rightarrow \infty} T^n \mathbf{x} = \mathbf{0} \quad (I - T)^{-1} = \lim_{n \rightarrow \infty} (I + T + T^2 + \cdots + T^n)$$

7. Give a sketch of a proof of the following, which is a generalization of a well known theorem in Calc I:

Let $f, f', \dots, f^{(n)}$ all be continuous on $[a, b]$, and suppose we have $n + 1$ roots of f in $[a, b]$. Then there exists $c \in (a, b)$ so that $f^{(n)}(c) = 0$.

SOLUTION: The “well known theorem” is Rolle’s theorem (or you could use the more general MVT). In the original version, we have: If f is continuous on $[a, b]$ and differentiable on (a, b) , and $f(a) = f(b)$, then there is a number c in (a, b) so that $f'(c) = 0$.

Generalizing once: If f has two roots and f is differentiable on (a, b) , then $f'(x)$ has one root in $[a, b]$.

Generalizing again: If f has three distinct roots, then $f'(x)$ has two distinct roots. If $f'(x)$ has two distinct roots and is continuous, then $f''(x)$ has one root.

Once more, we see that if f has four distinct roots, then f' will have three, and f'' will have two, and f''' will have one.

Continuing, we get the result: If $f, f', \dots, f^{(n)}$ are continuous on $[a, b]$, and f has $n + 1$ roots in $[a, b]$, then there exists $c \in (a, b)$ so that $f^{(n)}(c) = 0$

8. Let x_0, x_1, \dots, x_n be $n + 1$ real numbers, and let x be fixed (so its also constant). Consider the function of t :

$$\Psi(t) = \frac{(t - x_0)(t - x_1) \cdots (t - x_n)}{(x - x_0)(x - x_1) \cdots (x - x_n)}$$

(a) $\Psi(t)$ is a polynomial of what degree? SOLN: $n + 1$

(b) What is $\Psi(x_j)$, $j = 0, 1, 2, \dots, n$? What is $\Psi(x)$?

$\Psi(x_j) = 0$ because there will be a zero in the numerator. $\Psi(x) = 1$, because the numerator and denominator will be the same expression.

(c) What is the $n + 1$ st derivative of Ψ , with respect to t ?

It is a constant. Right now, the lead monomial of the polynomial is:

$$kt^{n+1}$$

where $k = 1/(x - x_0) \cdots (x - x_n)$. Taking the $n + 1$ derivatives will leave us only with $k(n + 1)!$

9. Prove the error formula for polynomial interpolation (you should use the previous exercises!):

Given $n + 1$ points,

$$(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n))$$

and suppose that $P(x)$ is the $(n^{\text{th}}$ degree) polynomial interpolating those points. Use the following:

$$g(t) = f(t) - P(t) - (f(x) - P(x))\Psi(t) \quad (1)$$

to show that there is a c so that

$$f(x) = P(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)\cdots(x-x_n)$$

Hint: Consider the $n + 1$ st derivative of g .

SOLUTION:

The function g given above has $n + 2$ zeros, since

$$g(x_j) = (y_j - y_j) - (f(x) - P(x))\Psi(x_j) = 0$$

for $j = 0, 1, 2, \dots, n$ and

$$g(x) = (f(x) - P(x)) - (f(x) - P(x))\Psi(x) = 0$$

Therefore, there is a c in the interval between the smallest x_j and the largest (we assume that x is in this interval as well) so that

$$g^{(n+1)}(c) = 0$$

Differentiate both sides of Equation 1 to get the final result:

$$g^{(n+1)}(c) = f^{(n+1)}(c) - P^{(n+1)}(c) - (f(x) - P(x))\Psi^{(n+1)}(c) =$$

$$f^{(n+1)}(c) - (f(x) - P(x)) \frac{(n+1)!}{(x-x_0)(x-x_1)\cdots(x-x_n)} = 0$$

so that

$$f(x) = P(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)\cdots(x-x_n)$$

10. Prove the “Main Theorem of Polynomial Interpolation”: Given $(x_1, y_1), \dots, (x_n, y_n)$ with distinct values of x , there is one and only one polynomial P of degree at most $n - 1$ that interpolates the data.

SET UP: We already know such a polynomial exists by the Lagrange formula (exists for distinct values of x), so we only need to prove the uniqueness. Start by assuming that there are two interpolating polynomials of degree $n - 1$ or less, $P(x)$ and $Q(x)$ and consider the polynomial: $P(x) - Q(x)$. How many zeros does $P - Q$ have?

SOLUTION: $P(x) - Q(x)$ has n roots, and is a polynomial of degree $n - 1$ or less. The only way that can happen is if $P - Q$ is the zero polynomial:

$$P(x) - Q(x) = 0 \quad \Rightarrow \quad P(x) = Q(x)$$

so in fact there is only one polynomial of degree $n - 1$ or less that will interpolate n points (with distinct values of x).

11. (Exercise 1, p. 181) Decide whether the following equations form a cubic spline:

$$S(x) = \begin{cases} x^3 + x - 1 & \text{if } 0 \leq x \leq 1 \\ -(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1 & \text{if } 1 < x \leq 2 \end{cases}$$

SOLUTION: The idea here is that you can go through the list of conditions for a cubic spline. Without any other information, we are just checking interior points for continuity of the original function and its two derivatives. We should find that S is continuous at $x = 1$ but $S'(x)$ is not continuous at $x = 1$.

12. What is the “not-a-knot” condition for the cubic spline?

SOLUTION: The third derivatives match at the second and the next to last knots. If they are labelled x_1, \dots, x_n , then:

$$S'''(x_2) = S'''(x_{n-1})$$

13. Find the linearization of F at $(u, v) = (1, 1)$:

$$F(u, v) = (u + e^{u-v}, 2u + v)$$

SOLUTION: (Side Remark: From class, recall that we need the linearization to get the formula for Newton’s Method)

We need the Jacobian matrix, DF . Then the linearization is:

$$L(u, v) = F(1, 1) + DF(1, 1)(u - 1, v - 1)^T$$

Continuing:

$$DF = \begin{bmatrix} 1 + e^{u-v} & -e^{u-v} \\ 2 & 1 \end{bmatrix}$$

Therefore,

$$L(u, v) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u - 1 \\ v - 1 \end{bmatrix}$$

14. Perform one step of Newton’s Method on the nonlinear system:

$$\begin{aligned} u^3 - v^3 + u &= 0 \\ u^2 + v^2 &= 1 \end{aligned}$$

beginning with the vector $(u, v) = (1, 1)$

SOLUTION:

$$F(1, 1) = [0, 1]^T \quad DF(1, 1) = \begin{bmatrix} 4 & -3 \\ 2 & 2 \end{bmatrix}$$

so that

$$\mathbf{x}_1 = \mathbf{x}_0 - DF^{-1}(1, 1)F(1, 1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{14} \begin{bmatrix} 2 & 3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 11 \\ 10 \end{bmatrix}$$

15. Give an example, or explain why it doesn’t exist:

- (a) A polynomial of degree 6 that is zero at $x = 1, 2, 3, 4, 5, 6$ and is 10 at $x = 7$.

SOLUTION: It might be true- there are 7 parameters to choose, and 7 conditions. We know that the polynomial must be of the form:

$$P(x) = k(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$$

Substitute $x = 7$, and $P(7) = 6!k = 10$, so $k = \frac{10}{6!}$. Yes, it is possible.

- (b) A polynomial of degree 6 that is zero at $x = 1, 2, 3, 4, 5, 6$ and is 10 at $x = 7$, and is 70 at $x = 8$.

SOLUTION: It might be true by chance, but in general we would have 8 conditions but only 7 parameters. We found the polynomial last time that satisfies the first 7 conditions. Checking, we have:

$$P(8) = \frac{10}{6!} 7! = 7 \cdot 10 = 70$$

so in this case, the answer is yes, there is such a polynomial.

16. Write the 7 Chebyshev points if we are to interpolate a function on the interval $[-1, 5]$.

SOLUTION:

The Chebyshev points in this case are:

$$\cos\left(\frac{\pi}{14}\right), \cos\left(\frac{3\pi}{14}\right), \cos\left(\frac{5\pi}{14}\right), \cos\left(\frac{7\pi}{14}\right), \cos\left(\frac{9\pi}{14}\right), \cos\left(\frac{11\pi}{14}\right), \cos\left(\frac{13\pi}{14}\right)$$

To put them in the interval from $[-1, 5]$, multiply by three and add two. Numerically, we end up with:

$$4.92 \quad 4.35 \quad 3.30 \quad 2 \quad 0.70 \quad -0.33 \quad -0.92$$

17. What is the worst error one can expect by using a polynomial of degree 5 to approximate the sine function on the interval $[0, 2]$ using the Chebyshev points?

SOLUTION: A degree 5 polynomial would mean that we are (generically) using 6 data points.

The error bounds are:

$$|f(x) - P_5(x)| \leq \frac{f^{(6)}(c)}{6!} \cdot \frac{1}{2^5} \cdot \left(\frac{2-0}{2}\right)^n = 4.34 \times 10^{-5} < 0.5 \times 10^{-5}$$

so we can expect accuracy to 5 decimal places (excellent!)