Homework: Vector Norms

How to Measure Distances Between Things in $\mathbb{R}^n$

Definition: The norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is a function from $\mathbb{R}^n$ to $\mathbb{R}$, denoted by $\|\cdot\|$, so that the following are properties are satisfied:

1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = \mathbf{0}$.
2. $\|c\mathbf{x}\| = |c|\|\mathbf{x}\|$, for all scalars c.
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (Triangle Inequality)

Definition: Given a vector space X and a norm $\|\cdot\|$, the distance between two vectors $\mathbf{x}, \mathbf{y}$ is $\|\mathbf{x} - \mathbf{y}\|$.

Definition: The $p-$norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is

$$\|\mathbf{x}\|_p = \left(|x_1|^p + |x_2|^p + \ldots + |x_n|^p\right)^{1/p}$$

In particular,

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n| \qquad \|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

And, we will define the "infinity norm" as:

$$\|\mathbf{x}\|_\infty = \max_i \{|x_i|\}$$

Exercise Set 1

1. Show that $\|\mathbf{x}\|_\infty$ satisfies the three parts in the definition of a norm.
2. What if two vectors were the same except for a single coordinate (without loss of generality, make it the first coordinate). What would the distance between them be under the $1-$, $2-$ and ∞ norms?
3. If two vectors (in $\mathbb{R}^n$) are within ϵ of each other in the ∞ norm, how close together are they in the $1-$ norm? (Hint: Start with definitions)
4. If two vectors (in $\mathbb{R}^n$) were within ϵ of each other in the $1-$ norm, how close together are they in the ∞ norm?
5. Let $F(\mathbf{c}) = F(c_1, c_2, \ldots, c_n) = c_1a_1 + c_2a_2 + \ldots + c_na_n$, where every $c_i \geq 0$, $\sum_i c_i = 1$, and $a_1, a_2, \ldots, a_n$ are given, fixed numbers. Find the maximum value of F, and the $\mathbf{c}$ where it occurs. (If you get stuck, try putting in some numbers for the a_i and simplifying the problem- For example, find the max of $3c_1 + 5c_2$)
6. Same function as before, except that the values of c_i are changed: This time, the only restriction is $|c_i| = 1$ for each i.