Homework: Matrix Norms

How to Measure Distances Between Things in $\mathbb{R}^n$

Definition: Given a vector space V, and a norm $\|\cdot\|$, we can define the induced matrix norm for a matrix A as:

$$\|A\| = \max_{\mathbf{x}\in\mathbb{R}^n} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

For example, every $p-$norm for $V = \mathbb{R}^n$ induces a matrix norm. In our work below, we will show that we can compute the induced matrix norm directly for some values of p. For the $2-$norm, we won't prove it, but if λ is the largest eigenvalue of A^TA, then $\|A\|_2 = \sqrt{\lambda}$.

We might also mention a matrix norm that is *not* an induced norm- It is the Frobenius norm:

$$\|A\|_{\text{fro}} = \left(\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}^2\right)^2$$

This is the norm you would get by treating the matrix A as one big vector, and using the 2-norm on that.

Exercise Set 2

You might find the solutions to Exercise Set 1 useful when you're solving these.

1. Show that $\|A\|_1$ =max abs col sum. HINT: Write $A\mathbf{x}$ in terms of the columns of A.
2. Show that $\|A\|_\infty$ =max abs row sum. HINT: Write $A\mathbf{x}$ in terms of the rows of A.
3. Find a 2×2 matrix A so that $\det(A) = 0$, but $\|A\|_p \neq 0$ for $p = 1, \infty$.
4. Show that for any induced matrix norm:
   (a) $\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$
   (b) $\|AB\| \leq \|A\|\|B\|$
   (c) $\|A+B\| \leq \|A\| + \|B\|$
5. Let λ_{min} be the smallest eigenvalue of A^TA, and let $B = A^{-1}$. Show that the largest eigenvalue of B^TB is $1/\lambda_{\text{min}}$
6. If we are solving $A\mathbf{x} = \mathbf{b}$, and $\mathbf{x}_c$ is our approximate solution (with $\mathbf{r} = \mathbf{b} - A\mathbf{x}_c$), show that the error magnification factor can be written as:

$$\frac{\|A^{-1}\mathbf{r}\|}{\|\mathbf{r}\|}\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$$