

## Numerical Analysis Final Exam

In this final exam, we will explore a single problem that spans much of what we discussed in class this semester. The problem, which you might have seen in Calculus Lab, is to determine the shortest path on the surface of the torus.

### 1. PRELIMINARIES

We will consider the torus defined by a small circle in the  $x - z$  plane of radius 1 centered at  $x = 2, y = 0, z = 0$ . To get the torus, we rotate this circle about the  $z$ -axis. If we define  $\alpha$  as the angle of rotation about the small circle, and  $\beta$  as the angle of rotation in the  $x - y$  plane, the equations of the torus are:

$$x = (2 + \cos(\alpha)) \cos(\beta), \quad y = (2 + \cos(\alpha)) \sin(\beta), \quad z = \sin(\alpha)$$

To visualize the torus in Maple, see the attached worksheet.

The equations describe  $x, y$ , and  $z$  in terms of  $\beta$  and  $\alpha$ , thus there is a 1-1 correspondence between points on the torus and an  $(\beta, \alpha)$  pair taken from  $[0, 2\pi) \times [0, 2\pi)$ . This rectangle we can refer to as the  $\beta, \alpha$  plane.

A path on the surface of the torus can be described by writing  $\alpha, \beta$  as functions of time,  $t$ , then substituting them into  $x, y, z$  which then defines a space curve ( $x, y, z$  will be functions of  $t$ ). In this case, if  $a \leq t \leq b$ , then the arc length of the path on the torus can be written as:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

As an example, consider the path defined by taking  $\beta = t, \alpha = -t(t - \pi)$ ,  $0 \leq t \leq \pi$ . This path will take us from  $\alpha = 0, \beta = 0$  (corresponding to  $(3, 0, 0)$  on the torus) to  $\beta = \pi, \alpha = 0$  (corresponding to  $(-3, 0, 0)$  on the torus). This parameterization leads to:

$$\begin{aligned} x &= (2 + \cos(-t(t - \pi))) \cos(t), & y &= (2 + \cos(-t(t - \pi))) \sin(t), \\ z &= \sin(-t(t - \pi)) \end{aligned}$$

See the Maple worksheet for the computation of the arc length.

### 2. THE GEODESIC

In general, we are told that the shortest distance between two points is taken by a straight line. This is true in the plane, but not necessarily on a curved surface (e.g., a sphere or a torus). What path does define the shortest length in such a situation? Something called the *geodesic*.

It turns out that, for a path to be shortest on the surface of a torus, it must satisfy a certain differential equation. If  $\alpha, \beta$  are the two angles as defined previously, and we further parameterize them in terms of  $t$ , then the shortest path is a solution to:

$$\begin{aligned} \alpha'' &= -(2 + \cos(\alpha)) \sin(\alpha) \cdot (\beta')^2 \\ \beta'' &= 2 \frac{\sin(\alpha)}{2 + \cos(\alpha)} \cdot \alpha' \cdot \beta' \end{aligned}$$

If we wish to traverse the torus beginning at a certain point and end at a certain point, then this becomes a 4-dimensional nonlinear boundary value problem with two unknowns:  $\alpha'(t_0)$  and  $\beta'(t_0)$ .

## 3. THE PROBLEM

Find the length of the shortest path between the points  $(3, 0, 0)$  and  $(-3, 0, 0)$ , corresponding to a path between  $(\beta, \alpha) = (0, 0)$  to  $(\beta, \alpha) = (\pi, 0)$ . This will involve the following steps:

- (1) Solve (approximately) the Boundary Value Problem. You might have to resort to some type of trial and error, but you may use Matlab's built-in `ode23s` function. Have Matlab return your solution at equally spaced points. A sample is attached.
- (2) The output to the BVP will be  $\alpha(t), \beta(t)$ , given pointwise. To compute the arc length, we will need to substitute these values into  $x, y, z$ , and then differentiate.

Compute  $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$  numerically.

- (3) Next we will need to approximate the arc length. Use the Composite Simpson's Rule that we put together in class.
- (4) Compare this answer using a different technique:  
 From the numerical approximation of  $\alpha, \beta$  from the BVP, use our linear algebra techniques to find an approximate degree 10 polynomial (that is, we'll be able to write  $\beta$  in terms of  $\alpha$ ). Use this polynomial in Maple to determine an approximate arc length (use the sample given as a guide).

When you're finished, turn in the following:

- (1) A write up explaining your technique in solving the BVP, and the initial values you ended up using to find  $\alpha, \beta$ . Attach a plot of  $\alpha$  and  $\beta$ , and state the error between your angles and the true angles  $(\pi, 0)$ .
- (2) A script or diary file for your derivative computations.
- (3) A script or diary file for your integration. Circle your answer for the shortest path!
- (4) A script or diary file for your best fitting curve (Use linear algebra, not the "Basic Fitting" tools!) You don't need to output the matrices, but you should show your final coefficients.
- (5) A Maple worksheet showing your arc length computations, and a plot of the curve on the torus. You can use the sample worksheet that can be downloaded from our class website.