# 1 Notes on Error Analysis

We want to look at the process of computation like a function that takes inputs and produces outputs; In error analysis, we are trying to analyze what happens when we introduce small changes to either the input or output. Errors in the input are called *backward error* and in the output are called *forward error.*

**Example 1:** If our problem is to evaluate a given function $f$ at a particular $x$, then backward error refers to changes in $x$, and forward error refers to changes in $f(x)$.
**Example 2:** If our problem is to find the root of a function $p$, then the backward error is the change in the function $p$ and the forward error is the change in the root.

It would be very nice to know how input errors (backward error) gets propagated through the function or algorithm into the output error (forward error). One way to quantify this relationship is by the *error magnification factor:*

$$\text{error magnification factor } = \frac{\text{relative forward error}}{\text{relative backward error}}$$

The **condition number** is the maximum error magnification factor taken over all changes in input.

Going back to our previous two examples:
**Example 1A:** Our problem here is to evaluate $f(x) = \sqrt{x}$ at some $x$. Then the relative backward error is:

$$\frac{|\Delta x|}{|x|}$$

and the relative forward error is:

$$\frac{|f(x + \Delta x) - f(x)|}{|f(x)|}$$

Putting these together for the error magnification factor:

$$\left( \frac{|f(x + \Delta x) - f(x)|}{|f(x)|} \right) / \left( \frac{|\Delta x|}{|x|} \right)$$

Re-writing and taking the limit as $\Delta x \to 0$, we get the condition number, $\kappa$ for the function evaluation problem:

$$\kappa = \frac{|f'(x)|}{|f(x)|/|x|}$$

So using $f(x) = \sqrt{x}$, we find $\kappa = 1/2$. Such a small number means that small changes in the input will generally correspond to small changes in the output (in fact, we might expect them to be a factor of 1/2).

**Example 2A:** Here the problem was in root-finding. In this case, assume that $p$ is a function and $r$ is its root ($p(r) = 0$). Assume that $r + \Delta r$ is the output of our root-finding algorithm. The backward error is

$$|p(r) - p(r + \Delta r)| = |p(r + \Delta r)|$$

and the forward error is $|\Delta r|$.

The backward error here might need some explanation: The backward error here is: "how do we change $p$ so that the new function has $r + \Delta r$ as its root?". We can translate this mathematically in a number of ways- For example, we might say that $r + \Delta r$ is the solution to $p(x) + \epsilon = 0$, or more generally as $p(x) + \epsilon g(x) = 0$.

We will use the more general form first- In that case, we will take $r$ to be the exact solution to $p(x) = 0$ and $r + \Delta r$ to be the *exact* solution to the perturbed function,

$$p(r + \Delta r) + \epsilon g(r + \Delta r) = 0$$

To get the error magnification, let's see if we can relate $\Delta r$ to $p'(r)$. To do this, perform a Taylor series expansion to the equation above,

$$p(r) + (\Delta r)p'(r) + \epsilon(g(r) + (\Delta r)g'(r)) + h.o.t. = 0$$

Solve for $\Delta r$ and ignore the higher order terms ($h.o.t.$)

$$\Delta r \approx \frac{-\epsilon g(r)}{p'(r) + \epsilon g'(r)} \leq \frac{-\epsilon g(r)}{p'(r)}$$

so that the error magnification factor is:

$$\left| \frac{\Delta r / r}{\epsilon g(r)/g(r)} \right| \approx \frac{|g(r)|}{|rp'(r)|}$$

## 1.1 Multiple Roots, Part I

To put the previous material into numbers, suppose we are looking at the roots to something very simple, like $p(x) = x^2 - 2x + 1 = (x-1)^2$ at $r = 1$. A small perturbation in the coefficients can lead to a larger change in the roots- For example,

$$x^2 - 2x + 0.9999 = (x - 0.99)(x - 1.01)$$

In this case, $\epsilon = -10^{-4}$, $g(x) = 1$. This led to an error of $10^{-2}$ in the roots. In fact, the condition number of this problem is infinite since $p'(r) = 0$.

## 1.2 Multiple Roots, Part II

Consider $f(x) = x^3 - 2x^2 + \frac{4}{3}x - \frac{8}{27}$. For future reference, note that this is simply the expansion of $f(x) = (x - 2/3)^3$. Also note that $f(2/3) = 0$, $f'(2/3) = 0$ and $f''(2/3) = 0$.

In Matlab, type:

```
fzero('x.^3-2*x.^2+4*x/3-8/27',1,optimset('Display','iter'))
```

After 64 iterations, Matlab stops with an answer where

$$|x^* - r| \approx 0.4 \times 10^{-5}$$

so that we only have 5 significant digits. In fact, Matlab thinks this is the exact solution ($f(x^*)$ is computed to be 0).

**Analysis of the Error:**

The problem here is that the function is too flat- In a small interval about $x = 2/3$, the IEEE arithmetic produces $y-$ values that are all zero.

We can view the process of finding a root of a function as equivalent to constructing a (local) inverse and evaluating the inverse at $y = 0$:

$$f(r) = 0 \Leftrightarrow r = f^{-1}(0)$$

This view is made explicit in Brent's Method, where we actually construct an approximation to the inverse using a quadratic function.

## 1.3 The Wilkinson Polynomial

A famous polynomial with roots that are hard to find numerically is the Wilkinson polynomial:

$$W(x) = (x-1)(x-2)(x-3)\cdots(x-20)$$

which is then expanded (an $m-$file with the polynomial expanded is on our class website, `wilkpoly.m` ). The problem here is that, when evaluated, we get cancellation by subtracting nearly equal numbers (and very large) numbers. For example,

```
wilkpoly(15)
ans =
-3.8085e+009
```

Let us consider what happens if we perturb the coefficient of $x^{15}$. What kind of change can we expect in the solution $r = 15$?

In this case, $g(x) \approx 1.67 \times 10^9 x^{15}$, and $W'(15) = 5!14!$, and the error magnification factor is:

$$\frac{|g(r)|}{|rW'(r)|} \approx \frac{1.67 \times 10^9 \times 15^{15}}{15 \cdot 4!15!} \approx 5.1 \times 10^{13}$$

The practical implication here is that we would not be surprised to lose 13 significant digits in computing the root.

### 1.3.1 Exercises:

1. Find the condition number for the problem: Evaluate the function $f(x) = x^2 + x$

2. Find the forward and backward error for the following functions, where $r = 3/4$ and the approximate root is 0.74:

   (a) $p(x) = 4x - 3$, (b) $p(x) = (4x-3)^2$, (c) $p(x) = (4x-3)^3$, (d) $p(x) = \sqrt[3]{4x-3}$

3. Let $p(x) = x^2 \sin(x)$ with $r = 0$.

   (a) Find the multiplicity of the root $r$.

   (b) Find the forward and backward error of the approximate root 0.01.