Quick Summary on Some ODEs

The purpose of these notes is to give some idea on how to solve some special first order differential equations.

1. Let $y' = f(t)$. Then $y(t) = \int f(t)\,dt + C$, where we look at $\int\,dt$ as "any particular antiderivative".

2. Let $y' = f(y)g(t)$. These are called separable differential equations. We can solve them in the following way:

$$\begin{aligned} \frac{dy}{dt} &= f(y)\,g(t) \\ \frac{1}{f(y)}\,dy &= g(t)\,dt \\ \int \frac{1}{f(y)}\,dy &= \int g(t)\,dt \end{aligned}$$

   EXAMPLE: Solve $y' = y(1-y)$, $y(0) = y_0$.

$$\frac{1}{y(1-y)}\,dy = 1\,dt \quad \Rightarrow \quad \int \frac{1}{y(1-y)}\,dy = \int 1\,dt$$

   The integral on the left is evaluated using Partial Fractions, where

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y} = \frac{1}{y} + \frac{1}{1-y}$$

   so we get:

$$\ln(y) - \ln(1-y) = t + C \quad \Rightarrow \quad \ln\left(\frac{y}{1-y}\right) = t + C \quad \Rightarrow \quad \frac{y}{1-y} = A\mathrm{e}^t$$

   where $A = \mathrm{e}^C$. Solve this for y:

$$y = \frac{A\mathrm{e}^t}{1 + A\mathrm{e}^t} = \frac{1}{1 + B\mathrm{e}^{-t}}$$

   We can put B in terms of y_0: $y(0) = y_0 = 1/(1+B)$ or $B = \frac{1-y_0}{y_0}$

3. Linear First Order: $y' - p(t)y = g(t)$

   To understand this method, note that:

$$\frac{d}{dt}\left(y(t)\mathrm{e}^{-\int p(t)\,dt}\right) = y'(t)\mathrm{e}^{-\int p(t)\,dt} + y(t)(-p(t)\mathrm{e}^{-\int p(t)\,dt} = \mathrm{e}^{-\int p(t)\,dt}\,(y' - p(t)y)$$

Therefore, multiplying both sides of the differential equation,

$$e^{-\int p(t)\,dt}\left(y' - p(t)y\right) = e^{-\int p(t)\,dt} g(t)$$

Simplifies to an equation like in Item 1 above:

$$\frac{d}{dt}\left(y e^{-\int p(t)\,dt}\right) = e^{-\int p(t)\,dt} g(t)$$

So that the general solution is:

$$y e^{-\int p(t)\,dt} = \int e^{-\int p(t)\,dt} g(t)\,dt$$

or,

$$y = e^{\int p(t)\,dt} \int e^{-\int p(t)\,dt} g(t)\,dt$$

EXAMPLE: Solve $y' = ty + t^3, \quad y(0) = y_0$.

First, $y' - ty = t^3$. The integrating factor: $e^{-\int t\,dt} = e^{-(t^2/2)}$. Continuing:

$$e^{-(t^2/2)}(y' - ty) = t^3 e^{-(t^2/2)} \quad \Rightarrow \quad \frac{d}{dt}\left(y e^{-(t^2/2)}\right) = t^3 e^{-(t^2/2)}$$

Integrate with respect to t (Use $u = \frac{1}{2}t^2$, $du = t\,dt$):

$$y e^{-(t^2/2)} = -(2 + t^2)e^{-(t^2/2)} + C \quad \Rightarrow \quad y = -(2 + t^2) + C e^{t^2/2}$$

Solve for C: $y(0) = y_0 = -2 + C \quad \Rightarrow \quad C = y_0 + 2$, and we get the final answer:

$$y(t) = -(2 + t^2) + (y_0 + 2)e^{t^2/2}$$

EXERCISES

1. Solve: $y' + 2ty = y + 4t - 2$, $y(0) = 1$
2. Solve: $y' = 3y$, $y(0) = 2$
3. Find value(s) of k for which the IVP:

$$ty' = 4y = 0, \qquad y(0) = k$$

has (i) No Solution, (ii) An Infinite Number of Solutions.