Homework: Assigned Monday, Sep 4
Due Friday, September 8

1. In Matlab, find the smallest value of p for which the expression:

$$\frac{\tan(x) - x}{x^3}$$

when evaluated in double precision at $x = 10^{-p}$ has no correct significant digits. (Hint: First we need the exact value- Take the limit (by hand) as $x \to 0$. Then let $p = -1, -2$, etc. until you find the right p value).

Solution: First we find the limit using L'Hospital's rule (by the way, did you check that L'Hospital was valid at each step?)

$$\lim_{x\to 0} \frac{\tan(x) - x}{x^3} = \lim_{x\to 0} \frac{\sec^2(x) - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^2(x)\tan(x)}{6x} =$$

$$\lim_{x\to 0} \frac{4\sec^2(x)\tan^2(x) + 2\sec^4(x)}{6} = \frac{1}{3}$$

In Matlab, we could type:

```
format long
x=10.^[-1 -2 -3 -4 -5 -6 -7 -8 -9];
y=(tan(x)-x)./(x.^3);  %Why is this ./ and .^?
```

You should see that the expression, evaluated at 10^{-8} returns 0.

2. Use the Bisection Method (in Matlab) to find the root to six correct decimal places: $3x^3 + x^2 = x + 5$.

Solution: We first need to rewrite this as $f(x) = 0$:

$$f(x) = 3x^3 + x^2 - x - 5$$

Now we need an initial interval, so we do a quick plot:

```
x=linspace(-3,3);
y=3*x.^3+x.^2-x-5;
plot(x,y)
```

In my case, I'll take the interval $[0, 2]$. Note that I can now predict how many iterations I'll need to get the right accuracy:

$$\frac{|b-a|}{2^{n+1}} \le \frac{1}{2} \times 10^{-6} \quad \Rightarrow \quad \frac{2}{2^{n+1}} \le \frac{1}{2} \times 10^{-6} \quad \Rightarrow \quad n = 21$$

Using the Bisection function from class, in Matlab I would type:

```
>> format long
>> f=inline('3*x.^3+x.^2-x-5');
>> y=bisect(f,0,2,0.5e-6);
Finished after 21 iterates
>> y
y =
   1.16972589492798
```

3. Use Bisection to find two real numbers x, correct to within 6 decimal places, that make the determinant of the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & x \\ 4 & 5 & x & 6 \\ 7 & x & 8 & 9 \\ x & 10 & 11 & 12 \end{bmatrix}$$

   equal to 1000. Hint: A little preprocessing in Maple might make this an easier problem.

   Solution: First, use Maple:

```
with(linalg):
A:=matrix(4,4,[1,2,3,x,4,5,x,6,7,x,8,9,x,10,11,12]);
f:=det(A);
                              2              4
                  -2475 - 202 x  + 1404 x + x
plot(f-1000,x=-20..15);
```

   We'll be using -3475 for our constant, since we're trying to find where the determinant is 1000.

   We see that the zeros are between $-20, -15$ and $5, 15$. Therefore, I'll use Matlab with intervals $[2, 4]$ and $[4, 6]$ (many choices possible):

```
format long
f=inline('x.^4-202*x.^2+1404*x-3475');
y=bisect(f,5,15,0.5e-6);
Finished after 24 iterates
y =
   9.70829933881760
y=bisect(f,-20,-15,0.5e-6);
Finished after 23 iterates
y =
 -17.18849807977676
```

4. You have a spherical tank with radius 1 meter. You pour 1 cubic meter of water into the tank. Find the height of the water to within 1 millimeter.

   Hint: The volume of the bottom H meters of a hemisphere of radius R is $\pi H^2(R - \frac{1}{3}H)$.

Solution: We are solving the following equation for H:

$$1 = \pi h^2 (1 - \frac{1}{3} h)$$

where we want h to be accurate to the thousandths place, or to within 0.0005. Make this into a root finding problem and find an initial interval (from physical conditions, we know the answer must be between $0, 1$, but we can plot it anyway):

```
>> format long
>> f=inline('pi*x.^2.*(1-(1/3)*x)-1');
>> x=linspace(0,1);
>> y=f(x);
>> plot(x,y);
>> y=bisect(f,0,1,0.0005);
Finished after 10 iterates
>> y
y =
   0.63525390625000
```

5. Use fixed point iteration to find a root of $\cos(x) = \sin(x)$. In order to compare your results with other people, let us get our iteration method by adding x to both sides, and start with $x_0 = 0$, with 19 iterations (before using Matlab, what is the solution?).

   Have Matlab compute e_{i+1}/e_i. Does this number look right? (Hint: Recall our proof of convergence).

   Solution: First we note that the solution is $\pi/4$. Next re-write the equation to a fixed point form. One way to do this would be to write:

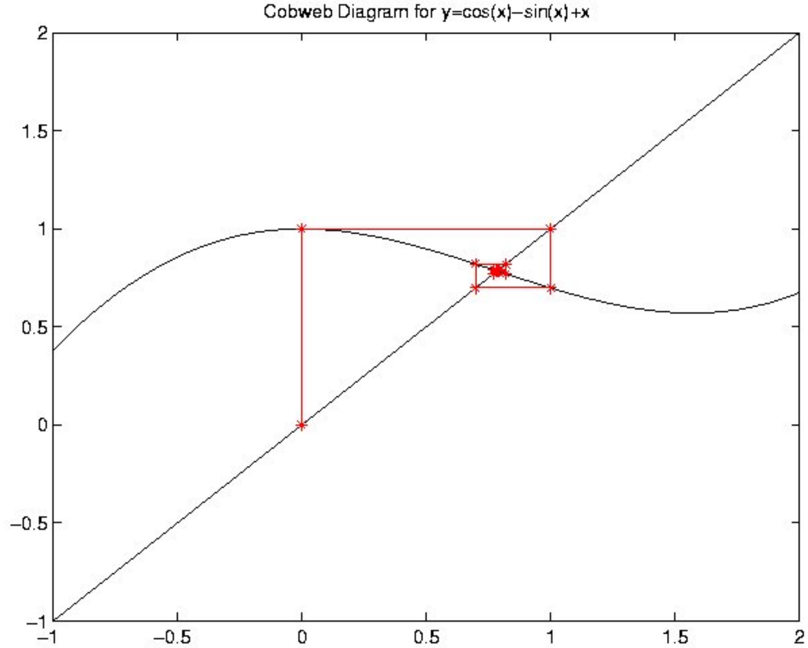
$$\cos(x) = \sin(x) \Leftrightarrow \cos(x) - \sin(x) + x = x$$

   Does this work? Let's be sure that the derivative is less than 1 (this wasn't a requirement, but it is a good idea):

$$f'(\pi/4) = -2/\sqrt{2} + 1 \approx -0.414...$$

   Now perform fixed point iteration:

```
>> f=inline('cos(x)-sin(x)+x');
>> y=fixpt(f,0,19);
>> format long
>> Error=abs(y-pi/4);
>> Z=Error(2:20)./Error(1:19);
>> Z'
ans =
   0.27323954473516
   0.40338351109981
   0.41244791418838
   0.41391310875287
   (middle terms deleted)
```




```
0.41421356224673
0.41421356252052
0.41421356222495
0.41421356323653
```

If we let $f(x) = \cos(x) - \sin(x) + x$, we see that the limit is $|f'(\pi/4)|$, as we used in our proof of convergence! That was,

$$|x_{n+1} - r| = |f(x_n) - f(r)| \leq |f'(c_n)||x_n - r| \quad \Rightarrow \quad \frac{|x_{n+1} - r|}{|x_n - r|} \leq |f'(c_n)| \to |f'(r)|$$