**Solutions to HW 3 that were due on Wed**
Done in Class

**EXERCISE 0.2.1 and Solution:** Prove that, if $g(r) = 0$ and $g$ is continuous on an interval about $r$, then there exists $\delta$ so that $|g(x)| \le k < 1$ for all $x$ in $(r - \delta, r + \delta)$. (HINT: Follow the proof of the Lemma)

**Solution:** Suppose there exists no such $\delta$. That means that for every $\delta$, there is a point $c$ in the interval $(r - \delta, r + \delta)$ so that $g(c) \ge 1$.
In that case, let $\delta_n = \left(\frac{1}{2}\right)^n$, and let $c_n$ be any point in $(r - \delta_n, r + \delta_n)$ so that $g(c_n) \ge 1$.
*Side Remark: Our choice of $\delta_n$ was somewhat arbitrary- Choose $\delta_n$ to be any sequence that goes to zero.*
Since $\delta_n \to 0$ as $n \to \infty$, then $c_n \to r$, and we have:

$$g(c_n) \ge 1 \text{ for all } n$$

so that

$$\lim_{n\to\infty} g(c_n) \ge 1 \text{ however } g(r) < 1$$

This violates continuity, which states that:

$$\lim_{n\to\infty} g(c_n) = g(r)$$

Therefore, there must exist a $\delta$ so that $|g(x)| \le k < 1$ for every $x$ in $(r - \delta, r + \delta)$.
*Side Remark: Why is k there? We did not want a sequence of points in $(r - \delta, r + \delta)$ so that the limit of the sequence was 1; in particular we did not want to say that $|g(x)| \le 1$.*
**EXERCISE 0.2.2 and Solution:** Give an interval on which we can guarantee the convergence of Newton's Method, if $g(x) = x^2 - 1$ (give the interval about the root $r = 1$).

**Solution:** First, let's look at $F(x)$ and $F'(x)$:

$$F(x) = \frac{x^2 + 1}{2x} \qquad F'(x) = \frac{1}{2}\left(1 - \frac{1}{x^2}\right)$$

So, $\delta_1$ is easy to get- $F$ is not continuous at $x = 0$, so we can make $\delta_1$ something like $1 - \epsilon$ (fill in the $\epsilon$ with something if you feel the need).
For $\delta_2$, we need that $|F'(x)| < 1$, or

$$\frac{1}{2}\left|1 - \frac{1}{x^2}\right| < 1 \Rightarrow \left|1 - \frac{1}{x^2}\right| < 2$$

If we look at the graph of $1 - 1/x^2$ for $x > 0$, we see that it is strictly increasing and has a horizontal asymptote at $y = 1$, and a vertical asymptote at $x = 0$. Therefore, we need to find $x$ where

$$1 - \frac{1}{x^2} = -2 \Rightarrow x = \frac{1}{\sqrt{3}}$$

Therefore, if $x > \frac{1}{\sqrt{3}}$, $F'(x) < 1$. We have found the maximum size of $\delta_2$, which would be the distance between $r = 1$ and $1/\sqrt{3}$, or:

$$\delta_2 = 1 - \frac{1}{\sqrt{3}} - \epsilon \approx 0.42265 - \epsilon$$

Conclusion: If $x_0 \in (0.57735, 1.42265)$, then Newton's Method will converge quadratically to $r = 1$.