# Homework 6 Solutions

Section 5.2, 1,2,3,7,10, Computer Problems 5.2.6, 7

1. 5.2.1: Apply the Composite Trapezoidal Rule with $m = 1, 2, 4$ panels to approximate the integral. Compute the error by comparing the correct value.

   (a) $\int_0^1 x^2\, dx = \frac{1}{3}$
   - $m = 1$, the area is 0.5. Error: 0.1666..
   - $m = 2$, the area is 0.375. Error: 0.04166..
   - $m = 4$, the area is 0.34375, Error: 0.0104166...

   (b) $\int_0^{\pi/2} \cos(x)\, dx = 1$
   - $m = 1$, the area is: 0.785398.., Error: 0.2146018..
   - $m = 2$, the area is: 0.948059.., Error: 0.0519405
   - $m = 4$, the area is: 0.987115.., Error: 0.012884

   (c) $\int_0^1 \mathrm{e}^x\, dx = \mathrm{e} - 1 \approx 1.7272$
   - $m = 1$, the area is: 1.85914.., Error: 0.140859
   - $m = 2$, the area is: 1.75391.., Error: 0.035649
   - $m = 4$, the area is: 1.72722.., Error: 0.008940

2. (5.2.2) Same integrals as above, but with the Composite Midpoint Rule.

   (a) $\int_0^1 x^2\, dx = \frac{1}{3}$
   - $m = 1$, the area is 0.25. Error: 0.0833..
   - $m = 2$, the area is 0.3125. Error: 0.020833..
   - $m = 4$, the area is 0.328125, Error: 0.00520833...

   (b) $\int_0^{\pi/2} \cos(x)\, dx = 1$
   - $m = 1$, the area is: 1.11072, Error: 0.11072
   - $m = 2$, the area is: 1.02617, Error: 0.026172
   - $m = 4$, the area is: 1.006454, Error: 0.006454

   (c) $\int_0^1 \mathrm{e}^x\, dx = \mathrm{e} - 1 \approx 1.7272$
   - $m = 1$, the area is: 1.64872.., Error: 0.06956
   - $m = 2$, the area is: 1.70051.., Error: 0.01776
   - $m = 4$, the area is: 1.71381.., Error: 0.0044665

3. (5.2.3) Same integrals as above, but with Composite Simpson's Rule.

(a) $\int_0^1 x^2\,dx = \frac{1}{3}$

- $m = 1$, the area is 1/3. Error: 0
- $m = 2$, the area is 1/3. Error: 0
- $m = 4$, the area is 1/3, Error: 0

(b) $\int_0^{\pi/2} \cos(x)\,dx = 1$

- $m = 1$, the area is: 1.002279, Error: 0.0022798
- $m = 2$, the area is: 1.0001345, Error: 0.0001345
- $m = 4$, the area is: 1.00000829, Error: 0.00000829

(c) $\int_0^1 \mathrm{e}^x\,dx = \mathrm{e} - 1 \approx 1.7272$

- $m = 1$, the area is: 1.718861, Error: 0.000579
- $m = 2$, the area is: 1.783188, Error: 0.0000370
- $m = 4$, the area is: 1.718284, Error: 0.00000232

4. Problem 5.2.7: Find the precision of the given formulas for $\int_{-1}^1 f(x)\,dx$. We note that this integral applied to $1, x, x^2, x^3, x^4$ is $2, 0, 2/3, 0, 2/5$, respectively.

   - $f(1) + f(-1)$: If $f(x) = 1$, we get an area of 2. If $f(x) = x$, we get an area of 0. If $f(x) = x^2$, we do not get 2/3. The degree of precision is therefore 1.
   - $(2/3)(f(-1) + f(0) + f(1))$. If $f(x) = 1$, we get an area of 2. If $f(x) = x$, we get 0. If $f(x) = x^2$, we get 4/3. The degree of precision is again 1.
   - $f(-1/\sqrt{3}) + f(1/\sqrt{3})$: If $f(x) = 1$, we get an area of 2. If $f(x) = x$, we get 0. If $f(x) = x^2$, we get 2/3. If $f(x) = x^3$, we get 0. If $f(x) = x^4$, we get 2/9. The degree of precision is 3.

5. Problem 5.2.10: Find $c_1, c_2, c_3$ so that the rule:

$$\int_0^1 f(x)\,dx \approx c_1 f(0) + c_2 f(1/2) + c_3 f(1)$$

has the highest degree of precision possible.

If $f(x) = 1$, the equation becomes:

$$1 = c_1 + c_2 + c_3$$

If $f(x) = x$, the equation becomes:

$$\frac{1}{2} = \frac{1}{2}c_2 + c_3$$

If $f(x) = x^2$, the equation becomes:

$$\frac{1}{3} = \frac{1}{4}c_2 + c_3$$

Solve the system of equations to get:

$$c_1 = \frac{1}{6} \qquad c_2 = \frac{2}{3} \qquad c_3 = \frac{1}{6}$$

Which is also Simpson's Rule.

6. Computer Problem 5.2.6

   Create a script file in Matlab:

```
%Script file for Computer Problem 5.2.7

f=inline('sqrt(1+(3*x.^2).^2)');
A=compsimp(f,0,1,32)

f=inline('sqrt(1+(sec(x)).^4)');
A=compsimp(f,0,pi/4,32)

f=inline('sqrt(1+(1./(1+x.^2)).^2)');
A=compsimp(f,0,1,32)
```

   whose output is:

```
A =
   1.54786565377914
A =
   1.27797806958318
A =
   1.27797805913564
```

   And where the Composite Simpson's Rule is:

```
function A=compsimp(f,a,b,m)
%Approximates the integral of f over [a,b] using 2m+1 points.

h=(b-a)/(2*m);
x=linspace(a,b,2*m+1);
y=f(x);
A1=4*sum(y(2:2:(2*m)));
A2=2*sum(y(3:2:(2*m-1)));
A=(h/3)*(y(1)+y(2*m+1)+A1+A2);
```

7. Computer Problem 5.2.7

For this problem, again type up a script file. This will use Composite Trapezoidal Rule, which looks like:

```
function A=comptrap(f,a,b,m)
%Approximates the integral of f over [a,b] using m+1 points.
h=(b-a)/m;
x=linspace(a,b,m+1);
y=f(x);

if m>1
    A=(h/2)*(y(1)+y(m+1)+2*sum(y(2:m)));
else
    A=(h/2)*(y(1)+y(2));
end
```

To call this function with

$$h = \frac{b-a}{1}, \quad \frac{b-a}{2}, \quad \frac{b-a}{2^2}, \ldots, \frac{b-a}{2^8}$$

note that in our function, $h = (b-a)/m$. Therefore, we will call this function for $m = 1, 2, 2^2, 2^3, \ldots, 2^8$.

What are we supposed to see? The Composite Trapezoidal Rule (as written on page 240) could be written as:

$$\int_a^b f(x)\,dx - \text{Trap}(h) = kh^2 \qquad \text{or} \qquad I - A(h) = kh^2$$

where $I$ is the actual area, $A$ is the approximate area (using the rule), $K$ is some constant, and $h = (b-a)/2^n$. Taking the log of both sides,

$$\ln|I - A| = 2\ln(h) + \ln(k)$$

so in the log-log plot, there should be a general linear shape with a slope of 2. There is a problem using the log-log plot, however- You can't directly use the "basic fitting" tools. Better to manually plot the log of the domain and range, as shown.

Once you get these basic point-plots, then you can determine the line of best fit using the "basic fitting tools". I did that, then went back into the code and had Matlab plot the lines with the data.

For easy reference, the script file is also included here for just $f(x) = x^2$.

```
f=inline('x.^2');
M=2.^[0:16];
k=length(M);
for j=1:k
    A(j)=comptrap(f,0,1,M(j));
end
figure(1)
x=log(1./M);
plot(x,log(abs((1/3)-A)),'r*',x,2*x-1.8,'b-')
title('Area for x^2')
```

where the code for the Composite Trapezoidal Rule is as given previously.