Homework Set 7: Chapter 6, Section 1

Due: Wednesday, November 8th

1. Do Exercises 1-4 on the "Quick Summary of Some ODEs" handout.

   (a) Exercise 1: First rewrite:

   $$y' + (2t-1)y = 4t-2$$

   And compute the "integrating factor", $e^{\int p(t)\,dt}$ (Note that the negative dropped, since $p(t)$ is added here). Next, multiply both sides of the equation by that, and integrate:

   $$\begin{aligned} e^{t^2-t}\left(y' + (2t-1)y\right) &= (4t-2)e^{t^2-t} \\ \left(ye^{t^2-t}\right)' &= (4t-2)e^{t^2-t} \\ ye^{t^2-t} &= \int (4t-2)e^{t^2-t}\,dt \end{aligned}$$

   To integrate, let $u = t^2 - t$, $du = (2t-1)\,dt$, so the integral above becomes $2\int e^u\,du$.

   $$ye^{t^2-t} = 2e^{t^2-t} + C \qquad y(t) = 2 + Ce^{-t^2+t}$$

   Solving for C, we get: $y(t) = 2 - e^{-t^2+t}$.

   (b) Exercise 2: There are a couple ways to get started. Here is one:

   $$\frac{1}{y}\,dy = 3\,dt \qquad ln|y| = 3t + C \qquad y = Ae^{3t}$$

   so if $y(0) = 2$, $A = 2$, and the solution is: $y(t) = 2e^{3t}$.

   (c) Exercise 3: Note the typo. Should be $ty' - 4y = 0$.

   $$y' - \frac{4}{t}y = 0$$

   Next, we get the integrating factor:

   $$e^{-4\int \frac{1}{t}\,dt} = e^{-4\ln|t|} = t^{-4}$$

   Multiply both sides of the ODE by this, and integrate:

   $$t^{-4}\left(y' - \frac{4}{t}y\right) = 0 \qquad (y\,t^{-4})' = 0 \qquad y\,t^{-4} = C$$

   so the general solution is $y(t) = Ct^4$

   Now going to the initial condition, we see that $y(0)$ must be zero. Therefore, if $k \neq 0$, there is no solution to the IVP. If $k = 0$, then C can be any constant, and we have an infinite number of solutions.

(d) Exercise 4: This is the Chain Rule on $f(t, y)$. If

$$\frac{dy}{dt} = f(t, y)$$

then

$$\frac{d}{dt}y' = f_t(t, y) + f_y(t, y)y'$$

so that:

$$\frac{d^2y}{dt^2} = f_t(t, y) + f(t, y)f_y(t, y)$$

2. Problem 6.1.7, p. 268 of the Chapter 6 handout.

   For which of these IVPs on $t \in [0, 1]$ does the E&U Theorem guarantee a unique solution? Find the Lipschitz constants if they exist:

   (a) $f(t, y) = t$. Therefore,

   $$|f(t, y_1) - f(t, y_2)| = |t - t| = 0$$

   This function is Lipschitz with Lipschitz constant 0. Therefore, the Existence and Uniqueness Theorem applies.

   (b) $f(t, y) = y$. Therefore,

   $$|f(t, y_1) - f(t, y_2)| = |y_1 - y_2|$$

   This function is Lipschitz with constant 1. Therefore, the Existence and Uniqueness Theorem applies.

   (c) $f(t, y) = -y$, so that:

   $$|f(t, y_1) - f(t, y_2)| = |-y_1 + y_2| = |y_1 - y_2|$$

   This function is again Lipschitz with constant 1, so the Existence and Uniqueness Theorem applies.

   (d) $f(t, y) = -y^3$. In this case,

   $$|f(t, y_1) - f(t, y_2)| = |-y_1^3 + y_2^3| = |y_1^3 - y_2^3| = |y_1^2 + y_1y_2 + y_2^2||y_1 - y_2|$$

   This term, $y_1^2 + 2y_1y_2 + y_2^2$ does not have a maximum, therefore this function is NOT Lipschitz. (Note that we might make it Lipschitz if we took y to be bounded).

3. Problem 6.1.9, p. 268 of the Chapter 6 handout.

   Find the solutions of the IVPs in the previous problem. For each equation, verify (if possible) Gronwall's Inequality if $Y(t)$ solves $y(0) = 0$ and $Z(t)$ solves $y(0) = 1$.

(a) If $y' = t$, then $y(t) = \frac{1}{2}t^2 + C$. Therefore, $Y(t) = \frac{1}{2}t^2$ and $Z(t) = \frac{1}{2}t^2 + 1$. In this case,

$$|Y(t) - Z(t)| = 1$$

for all t. This does satisfy Gronwall's Inequality:

$$|Y(t) - Z(t)| \le e^{0\cdot(t-0)}|0 - 1|$$

(b) If $y' = y$, then $y(t) = Ae^t$. Solving the two IVPs:

$$Y(t) = 0 \qquad Z(t) = e^t \quad \Rightarrow \quad |Y(t) - Z(t)| = e^t \le e^{1\cdot(t-0)}|0 - 1|$$

(c) Basically the same as in the previous equation, $y(t) = Ae^{-t}$:

$$Y(t) = 0 \qquad Z(t) = e^{-t} \quad \Rightarrow \quad |Y(t) - Z(t)| = e^{-t} \le e^{1\cdot(t-0)}|0 - 1| = e^t$$

(d) In this problem, we have: $y' = -y^3$, and to solve:

$$\int y^{-3}\,dy = \int -1\,dt \quad \Rightarrow \quad -\frac{1}{2}\cdot\frac{1}{y^2} = -t + C \quad \Rightarrow \quad y^2 = \frac{1}{C + 2t}$$

At this point, should we take the positive or negative root? Try both and see that the positive root does not satisfy the ODE (Off by the negative sign). Therefore, the general solution is:

$$y = (C + 2t)^{-1/2} = \frac{1}{\sqrt{C + 2t}} \qquad \text{with} \qquad y' = -(C + 2t)^{-3/2}$$

Now, if $y(0) = 0$, we see that no value of C will work- However, going back to the original ODE, we see that $y(t) = 0$ is a solution. Therefore,

$$Y(t) = 0$$

For $Z(t)$, $C = 1$, and: $Z(t) = \frac{1}{\sqrt{1+2t}}$. Notice that $|Z(t)| \to \infty$ as $t \to -1/2$.

4. Computer Problem 6.1.4. Only do parts (a, c, f)- We'll do the actual solutions in class.

   Here is a sample Script File and Euler's Method function that will produce the three plots:

```
%Script file for Computer Problem 6.1.4.   Here, note that we can
%  index the functions and the solutions:
DE{1}=inline('t','t','y');
DE{2}=inline('2*(t+1).*y','t','y');
DE{3}=inline('(t.^3)./(y.^2)','t','y');
```

```
Actual{1}=inline('(1/2)*t.^2+1')
Actual{2}=inline('exp(t.^2+2.*t)');
Actual{3}=inline('((3/4)*t.^4+1).^(1/3)');
M=10*2.^[0:8];

for k=1:3
    for j=1:length(M)
        [t,y]=euler(DE{k},[0,1],1,M(j));
        h(j)=t(2)-t(1);
        A(k,j)=y(end);
        clear t y
    end
end

for k=1:3
    subplot(2,2,k)
    plot(log(h),log(abs(Actual{k}(1)-A(k,:))),'r*-');
end

% **************Sample Euler: ***************
function [t,y]=euler(F,interval,y0,m)
t=linspace(interval(1),interval(2),m+1);
h=t(2)-t(1);
y=zeros(m+1,1);
y(1)=y0;

for j=1:m
    y(j+1)=y(j)+h*F(t(j),y(j));
end
```