In-Class Problems, Section 3.4-3.5

1. Let $F(c_1, c_2, \ldots, c_n) = c_1a_1 + c_2a_2 + \ldots + c_na_n$, where every $c_i \geq 0$, $\sum_i c_i = 1$, and $a_1, a_2, \ldots, a_n$ are given, fixed numbers. Find the maximum value of F.

2. Same function as before, except that the values of c_i are such that the only restriction is $|c_i| = 1$ for each i.

3. Show that if a is a real number, and $|a| < 1$, then:

$$\sum_{j=0}^{\infty} a^j = \frac{1}{1-a}$$

   Hint: Expand and simplify: $(1-a)(1+a+a^2+a^3+\ldots+a^n)$

4. Let A be an $m \times n$ matrix, and $x \in \mathbb{R}^n$. How would we write $A\mathbf{x}$ in terms of the columns of A? How would we write $A\mathbf{x}$ in terms of the rows of A?

5. Show (using the definition) that: $\|A\|_1 =$ maximum absolute column sum

   Hint: Write $A\mathbf{x}$ in terms of the columns of A.

6. Show (using the definition) that $\|A\|_\infty =$ maximum absolute row sum

   Hint: Write $A\mathbf{x}$ in terms of the rows of A

Definition: Given a norm for vectors, $\|\mathbf{x}\|$, we defined the norm of a matrix:

$$\|A\| = \max_{\mathbf{x} \in \mathbb{R}^n} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

7. Show that for any induced matrix norm, $\|A\mathbf{x}\| \leq \|A\| \, \|\mathbf{x}\|$

8. Let A, B be matrices. Show that for any induced matrix norm, $\|AB\| \leq \|A\| \, \|B\|$

9. Show that given an induced matrix norm, $\|A\| = 0$ iff A is the zero matrix.

10. Show that given an induced matrix norm, $\|A + B\| \leq \|A\| + \|B\|$

11. Construct a 2×2 matrix A so that $\det(A) = 0$, but $\|A\|_p \neq 0$ for $p = 1, \infty$.

Definition:[1] Given a matrix norm, the condition number for a matrix A is:

$$\text{cond}(A) = \|A\| \, \|A^{-1}\|$$

[1] There is a more general definition for the condition number of a function, but this is the accepted definition for the system $A\mathbf{x} = \mathbf{b}$

12. Show that for any induced matrix norm, $\text{cond}(AB) \le \text{cond}(A)\text{cond}(B)$

13. If A is a square matrix with $\|A\| < 1$, then $\|(I-A)^{-1}\| \le \dfrac{1}{1-\|A\|}$

    Hint: Look at Problem 3

14. Show that if A is invertible, and λ is an eval of A, then $\frac{1}{\lambda}$ is an eval of A^{-1}.

    Definition: $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T\mathbf{x}}$

15. Show that, if λ is an eigenvalue of A^TA, then $\lambda \ge 0$ (HINT: Compute $\|A\mathbf{v}\|_2^2$, where $\mathbf{v}$ is the eigenvector associated with λ)

16. Recall that our book stated that $\|A\|_2 = \sqrt{\lambda_{\max}}$, where $\lambda_{\max}$ is the largest eigenvalue of A^TA. Show that $\|A^{-1}\|_2 = \sqrt{\frac{1}{\lambda_{\min}}}$, where $\lambda_{\min}$ is the smallest eigenvalue of A^TA.

    Note that this shows, using the 2−norm, $\text{cond}(A) = \sqrt{\dfrac{\lambda_{\max}}{\lambda_{\min}}}$

17. Show that, using the 2−norm, $\text{cond}(A^TA) = (\text{cond}(A))^2$

    What does this imply about solving the normal equation associated with $A\mathbf{x} = \mathbf{b}$.

18. Is the following a "norm" for a matrix A? Why or why not?

    $$\|A\| \stackrel{?}{=} |\det(A)|$$

19. Condition numbers are used tell us if small changes can produce large changes. In particular, let $A\mathbf{x} = \mathbf{b}$ and $A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$, where $\tilde{\mathbf{x}} = \mathbf{x} + \Delta\mathbf{x}$ and $\tilde{\mathbf{b}} = \mathbf{b} + \Delta\mathbf{b}$. Show that:

    $$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} \le \text{cond}(A)\frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|}$$

    (Hint: Start with $\Delta x = A^{-1}\Delta b$) Similarly,

    $$\frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|} \le \text{cond}(A)\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|}$$

    Verify your results by looking at the sample computation on our class webpage (under the Chapter 3 Links).

The rule of thumb for condition numbers: **Expect to lose** $\log_{10}(\text{cond}(A))$ **significant digits in solving** $A\mathbf{x} = \mathbf{b}$
