In Class Problems: 4.1-4.4

1. Fill in the gaps in the proof of Theorem 4.2. Additionally, show that:

$$\frac{1}{n!}\int_s^x (x-t)^n f^{n+1}(t);, dt = \frac{1}{n+1!} f^{n+1}(\alpha)(x-s)^{n+1}$$

   This shows that you can use either remainder formula.

2. (Problem 9, p. 61. You may use Maple for the diff/ints) Find the first three terms of the Taylor expansion of $1/(1+x^2)$ about $x=0$. On the interval from $[-1,1]$, estimate the error using the two Taylor formulas, then compute the actual error. Are the estimates reasonable?

3. (Error on Lagrange Polynomials) Suppose we are approximating f on $x_0, x_1, \ldots, x_n$ which are in the interval $[a,b]$. Let P be the Lagrange polynomial (which is degree n). Let x^* be a fixed value of x (not equal to any x_i) in the interval $[a,b]$. Consider the function $g(t)$ given by:

$$g(t) = f(t) - P(t) - [f(x^*) - P(x^*)]\frac{(t-x_0)(t-x_1)\cdots(t-x_n)}{(x^*-x_0)(x^*-x_1)\cdots(x^*-x_n)}$$

   (a) Show that g has $n+2$ roots.

   (b) This implies that $g^{n+1}(\xi)=0$ for some $\xi \in [a,b]$ (Why?)

   (c) Show that

$$\frac{d^{n+1}}{dt^{n+1}}\left[\frac{(t-x_0)(t-x_1)\cdots(t-x_n)}{(x^*-x_0)(x^*-x_1)\cdots(x^*-x_n)}\right] = \frac{(n+1)!}{(x^*-x_0)(x^*-x_1)\cdots(x^*-x_n)}$$

   HINT: Don't do this directly- Can you write a lemma about taking the k^{th} derivative of a degree k polynomial?

   (d) Put the previous items together to prove Theorem 4.3:

$$f(x) - P(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

4. (Newton's Divided Differences: Be sure to read the text first, p. 64, 65). The form of the polynomial is:

$$P(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1) + \ldots + c_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

   We could get the formula for divided differences by considering:

$$f(x_0) = P(x_0) = c_0$$

$$f(x_1) = P(x_1) = f(x_0) + c_1(x_1-x_0) \qquad \Rightarrow \qquad c_1 = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

   and so on, but the table is very convenient.

(a) Compute the divided difference polynomial using a table for the data:

x	-2	-1	0	1	2
y	-1	3	1	-1	3

(b) Show that $f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$. Hint 1: $c_n = f[x_0, x_1, \ldots, x_n]$. Hint 2: Consider $g(x) = f(x) - P_n(x)$ and do something similar to Problem 5(b)

5. (Problem 12, p. 61) Extend the Weierstrass Theorem to show that, if f is differentiable on $[a, b]$, then for any $\epsilon > 0$, there exists a polynomial $p_n(x)$ such that:

$$\max_{a \leq x \leq b} |f(x) - p_n(x)| < \epsilon \qquad \text{and} \qquad \max_{a \leq x \leq b} |f'(x) - p_n'(x)| < \epsilon$$

This ensures that both *position* and *velocity* are well represented- Important in physics!

6. How would we actually compute the polynomial from the above problem? Given $n+1$ points, we could construct a polynomial like this:

$$H(x) = C + C_{0,1}(x - x_0) + C_{0,2}(x - x_0)^2 + C_{1,1}(x - x_0)^2(x - x_1) + C_{1,2}(x - x_0)^2(x - x_1)^2 +$$
$$C_{2,1}(x - x_0)^2(x - x_1)^2(x - x_2) + C_{2,2}(x - x_0)^2(x - x_1)^2(x - x_2)^2 + \ldots$$
$$+ C_{n,1}(x - x_0)^2(x - x_1)^2 \ldots (x - x_{n-1})^2(x - x_n)$$

(a) What is the degree of H?

(b) How many unknown coefficients are there? How many conditions must H satisfy?

(c) What happens if $x_0 = x_1 = x_2 = \ldots = x_n$?

(d) What form do you think H would take if we did not require the derivatives to also be interpolated?

(e) Using the model and our requirements, we could write a system of linear equations:

$$\begin{aligned} f(x_0) &= C \\ f'(x_0) &= C_{0,1} \\ f(x_1) &= C + C_{0,1}(x_1 - x_0) + C_{0,2}(x_1 - x_0)^2 \\ f'(x_1) &= C_{0,1} + 2C_{0,2}(x_1 - x_0) + C_{1,1} \\ &\vdots \end{aligned}$$

Continue these equations until you see a pattern. The polynomial considered here is called the *Hermite interpolating polynomial.*

7. Using the previous problem as a template, how would we construct a polynomial $p(x)$ so that the polynomial agrees with $f(x)$ at x_0, x_1, and $f'(x_0) = p'(x_0)$ and $f''(x_0) = p''(x_0)$? Find this polynomial if $f(x) = \mathrm{e}^{-x}\cos(x)$, and $x_0 = 0$, $x_1 = 1$, and verify your results.