Numerical Analysis Questions: April 15th, 2005

1. Empirically determining the rate of convergence.

   (a) Show empirically that the sequence $x_n = \frac{1}{n^3}$ has only linear convergence.

   (b) Show empirically that the sequence $x_n = 10^{-2^n}$ has quadratic convergence. HINT: In this case, you might analytically compute the logarithms before the computation to get better results.

   (c) (Analytic question) What is the order of convergence of the sequence 10^{-A^n}, where $A > 0$?

   (d) For each of the following, obtain an empirical estimate of the order of convergence of the sequence of iterates the method produces.

      i. $f(x) = x^2 - 2$, $[0, 2]$ with Bisection.
      ii. $f(x) = x^2 - 2$, $[0, 2]$ with Method of False Position.
      iii. $f(x) = (x - 2)^3$, $x_0 = 4$, Newton's Method.
      iv. $f(x) = (x - 2)^2$, $x_0 = 4$, Newton's Method.
      v. $f(x) = x(x - 2) = x^2 - 2x$, $x_0 = 4$, Newton's Method.

   CODING NOTES: Code up each algorithm so that you can pass the method an `inline` function (and its derivative if needed). Your algorithm should use a maximum number of iterations and a check on function tolerance. The function tolerance can be hard-coded as something like `1e-14` so that it doesn't need to be passed in as an argument. For example:

```
ftol=1e-14;
for j=1:maxiters

     statement;
     statement;

     if something < ftol
        fprintf('Algorithm finished on iterate %d\n',j);
        break
     end

     statement;
     statement;
  end
```

   Finally, note that we want the function to output all of the iterates of the algorithm, not just the final result.

2. Complex Newton's Method

   Newton's method can be applied to complex functions. In particular, consider:

$$f(z) = z^3 + az + b$$

   where a, b are complex coefficients. There are three roots (counting multiplicity) for f. The question we want to ask is this: Given an arbitrary initial point, z_0 in the complex plane, is it possible to predict where the sequence of iterates (produced by Newton's Method) will converge?

   The idea behind our code is simple: Given each z_0 in the complex plane, run Newton's Method until it converges to one of the three roots. Color **the initial point** z_0 by one of three colors, one color for each root. We can therefore, in theory, color the entire complex plane- Each color corresponding to the root for which Newton's Method will converge. The resulting plot is called a plot of the *basins of attraction* for Newton's Method. That is, a point that is colored blue will be attracted to the blue root. A point that is colored red will be attracted to the red root, etc.

   Before you start: Think about what the coloring might look like. Around each root, for example, the coloring should be uniform (these initial points should converge rapidly to their corresponding roots).

   Here's the pseudo-code for the algorithm:

```
Fix a, b for f(z)
Choose z0 from the complex plane.
Run Newton's Method:
  If zn converged to a root, color z0 by which root

Repeat for a new z0
```

   The coding is straightforward, but can take a while. A sample script file is on our class website that you can use (be sure to read the script file carefully, so you understand what it is doing!) Here are some interesting examples to consider:

   (a) $f(z) = z^3 - 1$ The roots are the "three roots of unity"
   (b) $f(z) = z^3 - 1.5z$

   Change the code so that you'll input three roots, a, b, c. This way you won't have to call `roots`, and you can use *polyval* for the function and derivative calls. Using this code, find the basins of attraction if $a = 1$, $b = -1.384609 - 0.9i$, $c = 0.384609 + 0.93i$. If you're feeling adventurous, change the code so that you've zoomed into $x \in [-0.17, 0.17], y \in [[-0.17, 0.17]$ (This might take Matlab a while longer to compute- Be patient!)

Can we answer the original question: Given an arbitrary initial point, z_0 in the complex plane, is it possible to predict where the sequence of iterates (produced by Newton's Method) will converge?

(Side remark: If you want to keep your pictures, Matlab can export the figures you produce to JPG format. In that case, you might remove the `colorbar` command from the script file.)