M467 Lab, Section 3.5 (Feb 4)

Before working with Matlab:

1. A "norm" is a special function from a vector space to the real numbers. Write the mathematical equivalents of the following:
   (a) The norm must be nonnegative. The norm is zero if and only if the vector is zero.
   (b) Something about the norm of a scalar multiple of a vector
   (c) The triangle inequality

2. The norms we will work with are the $1-$, $2-$, and $\infty-$ norms.
   (a) Compute the $1-$, $2-$ and ∞ norms for $\mathbf{x} = (1, 0, -1, 2)^T$.
   (b) Compute the distance between the vectors $\mathbf{x}$ and $\mathbf{y}$ under the $1-$, $2-$ and ∞ norms: $\mathbf{x} = (-1, 0, 1, 0)^T$ and $\mathbf{y} = (0, 1, 1, 1)^T$

3. The formal definition of the induced matrix norm:

   Given the vector space $\mathbb{R}^n$, and given a norm $\|\cdot\|$, and a matrix A, the matrix norm is defined to be:
   $$\|A\| = \max_{\mathbf{x}\in\mathbb{R}^n} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$
   This is too complicated to compute numerically. We will show later that:

   - $\|A\|_1 =$ Max abs col sum $= \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$
   - $\|A\|_2 = \sqrt{\lambda_{\max}}$, where λ's are the eigenvalues of $A^T A$.
   - $\|A\|_\infty =$ Max abs row sum $= \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$

4. Here is a matrix norm that is *not* induced by a vector norm. It is called the Frobenius[1] norm, and it comes up when a matrix A is treated as a "vector" in $\mathbb{R}^{mn}$:
   $$\|A\|_F = \sqrt{\sum_i \sum_j a_{ij}^2}$$

5. In Matlab, matrix and vector norms are called by the same command:

```
norm(x,p); %p-norm of the vector x (p>=1)
norm(A,1); norm(A,2); norm(A, inf); norm(A,'fro');  %Matrix norms
```

[1]Ferdinand Georg Frobenius was a German mathematician in the last half of the 1800's. He was "occasionally choleric, quarrelsome, and given to invectives", but they say that of a lot of the German mathematicians of that time. (http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Frobenius.html)

Lab Questions:

1. (Problem 6, p. 45) A parachutist jumps from a plane and the distance of his drop is measured. Suppose that the distance of descent s as a function of time t can be modeled by:

$$s = at + bt^2 e^{-0.1t}$$

Find values of a, b that are reasonable given the data below:

t	5	10	15	20	25	30
s	30	83	126	157	169	190

Plot the resulting model over time, showing the original data points as well.

2. Matlab uses the following: $\|A\|_F$=`sqrt(sum(diag(X'*X)))`

Where `diag(A)` returns a vector with the diagonal elements of A, `sum(x)` sums the elements of a vector, and `sqrt` is the square root. Show that this does give a valid computation of the Frobenius norm. You might start with a 3×3 matrix.

3. Use Maple to plot the set of points in $\mathbb{R}^2$ where $\|(x, y)^T\|_p = 1$, for $p = 1, 2, 10, 100$. Here's how to plot $\|x\|_1 = 1$ and $\|x\|_2 = 1$:

```
with(plots):
d1 := abs(x)+abs(y) = 1;
d2:=(x^2+y^2)^(1/2)=1;
A:=implicitplot(d1,x=-1.5..1.5, y=-1.5..1.5,grid=[30,30]):
B:=implicitplot(d2,x=-1.5..1.5, y=-1.5..1.5,grid=[30,30],color=black):
display({A,B});
```

Notice how the sets of points are nested, and as $p \to \infty$, the limit seems to be $\|x\|_\infty$.

4. In Maple, construct a graph of the function

$$\mathbf{x} \in \mathbb{R}^2 \to \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p}, \text{ where } A = \begin{bmatrix} -0.54 & -1 \\ -2 & 3 \end{bmatrix}$$

in the unit square. Estimate $\|A\|_1, \|A\|_2$, and $\|A\|_{100}$ from your graphs, and compare with the actual values (use $\|A\|_\infty$ for $\|A\|_{100}$).

5. What does it mean to say that a matrix is close to being singular?

6. The **condition number** of a matrix A is defined to be:

$$\text{cond}(A) = \|A\| \, \|A^{-1}\| \qquad \text{in Matlab: } \texttt{cond(A,p)}$$

It is said that, in general, matrices with large condition numbers should be avoided (that is, large condition numbers can lead to numerical errors in solving the associated matrix equation).

In using the induced $p-$norms, the condition number will always be greater than 1. So what matrices have the "ideal" condition number? It turns out that orthogonal[2] matrices have the perfect condition number.

Verify the statement above in Matlab for a few random orthogonal matrices, and for various dimensions. You can get a "random" orthogonal matrix in the following way:

```
A=randn(3,3);
[Q,S,V]=svd(A);  %Q is an orthogonal matrix- We won't use S, V
```

Since the determinant of an orthogonal matrix is 1, we might be led to think that the condition number is related to the determinant. Unfortunately, it isn't. Try the following:

(a) Let the matrix B_n be an $n \times n$ matrix defined as having all $1's$ along its diagonal, zeros below the diagonal, and -1's above the diagonal.

- Have Matlab compute the inverse of B, for $n = 2, 3, 4$.
- From this, what will $\|B\|_\infty$ be? What will $\|B^{-1}\|_\infty$ be?
- What is the determinant of B?

Purpose of this question: The determinant can be small, but the condition number can grow arbitrarily large.

(b) Define $D_n = \text{diag}(10^{-1}, 10^{-1}, \ldots, 10^{-1})$

- What is the determinant of D_n?
- What will the condition number of D_n be under the $1, 2$ or ∞ norms?

Purpose of this question: The determinant of the matrix can go to zero (singular), and still have a good condition number.

So if the condition number is not a measure of "how invertible" a matrix is, what is it a measure of?

The condition number is like a derivative in the sense that it tells us how stable our results are. That is, if we have the equation $A\mathbf{x} = \mathbf{b}$, and $\mathbf{b}$ is changed very slightly, how much change is there in our solution? If the condition number is large, there may be a substantial change in the solution.

[2]A matrix that is orthogonal is square with orthonormal columns