Numerical Analysis Final Exam

In this final exam, we will explore a single problem that spans much of what we discussed in class this semester. The problem, which you might have seen in Calculus Lab, is to determine the shortest path on the surface of the torus.

1. Preliminaries

We will consider the torus defined by a small circle in the $x-z$ plane of radius 1 centered at $x = 2, y = 0, z = 0$. To get the torus, we rotate this circle about the $z-$axis. If we define α as the angle of rotation about the small circle, and β as the angle of rotation in the $x-y$ plane, the equations of the torus are:

$$x = (2 + \cos(\alpha))\cos(\beta), \quad y = (2 + \cos(\alpha))\sin(\beta), \quad z = \sin(\alpha)$$

To visualize the torus in Maple, see the attached worksheet.

The equations describe x, y, and z in terms of β and α, thus there is a 1-1 correspondence between points on the torus and an (β, α) pair taken from $[0, 2\pi) \times [0, 2\pi)$. This rectangle we can refer to as the β, α plane.

A path on the surface of the torus can be described by writing α, β as functions of time, t, then substituting them into x, y, z which then defines a space curve (x, y, z will be functions of t). In this case, if $a \le t \le b$, then the arc length of the path on the torus can be written as:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

As an example, consider the path defined by taking $\beta = t$, $\alpha = -t(t - \pi)$, $0 \le t \le \pi$. This path will take us from $\alpha = 0, \beta = 0$ (corresponding to $(3, 0, 0)$ on the torus) to $\beta = \pi, \alpha = 0$ (corresponding to $(-3, 0, 0)$ on the torus). This parameterization leads to:

$$x = (2 + \cos(-t(t - \pi)))\cos(t), \quad y = (2 + \cos(-t(t - \pi)))\sin(t),$$

$$z = \sin(-t(t - \pi))$$

See the Maple worksheet for the computation of the arc length.

2. The Geodesic

In general, we are told that the shortest distance between two points is taken by a straight line. This is true in the plane, but not necessarily on a curved surface (e.g., a sphere or a torus). What path does define the shortest length in such a situation? Something called the *geodesic.*

It turns out that, for a path to be shortest on the surface of a torus, it must satisfy a certain differential equation. If α, β are the two angles as defined previously, and we further parameterize them in terms of t, then the shortest path is a solution to:

$$\alpha'' = -(2 + \cos(\alpha))\sin(\alpha) \cdot (\beta')^2$$

$$\beta'' = 2\frac{\sin(\alpha)}{2 + \cos(\alpha)} \cdot \alpha' \cdot \beta'$$

If we wish to traverse the torus beginning at a certain point and end at a certain point, then this becomes a 4-dimensional nonlinear boundary value problem with two unknowns: $\alpha'(t_0)$ and $\beta'(t_0)$.

3. The Problem

Find the length of the shortest path between the points $(3, 0, 0)$ and $(-3, 0, 0)$, corresponding to a path between $(\beta, \alpha) = (0, 0)$ to $(\beta, \alpha) = (\pi, 0)$. This will involve the following steps:

(1) Get the optimal α, β. Some notes:
   - The two second order equations should be written as a system of 4 first order equations. You can use the M-file on the website `TorusDE.m`, which we also wrote in class.
   - For each vector of initial conditions (the unknowns are $\alpha'(0), \beta'(0)$), use `ode23s` to output the values of $\alpha, \alpha', \beta, \beta'$ at time 1.
   - The goal is to determine $\alpha'(0), \beta'(0)$ so that $\alpha(1) = 0$ and $\beta(1) = \pi$. You'll need to write an error function that you can use with `Amoeba`.

   Here's a suggestion for the functions:

   `Amoeba` $\rightarrow$ Error Function $\rightarrow$ `ode23s`

   where `Amoeba` calls `ErrorFunction` using p initial conditions. `ErrorFunction` calls `ode23s` p times and outputs a vector of p errors. The Matlab script on our website gives you a sample of how to call `ode23s` to get one path.

(2) The output to the BVP will be $\alpha(t), \beta(t)$, given pointwise. To compute the arc length, we will need to substitute these values into x, y, z, and then differentiate.

   Compute $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ numerically. You may use the fact that `ode23s` gives α', β', but you may ignore that as well.

(3) Next we will need to approximate the arc length integral. Consider your choices in the write up, and choose one. Be sure to circle your answer.

(4) Compare this answer using a different technique:
Find a degree 10 polynomial that approximates the integrand, and compute the integral using it (use linear algebra and `polyval`).

When you're finished, turn in the following:

(1) A write up explaining your technique in solving the BVP, and the initial values you ended up using to find α, β. Attach a plot of α and β, and state the error between your angles and the true angles $(\pi, 0)$.
(2) A script or diary file for your derivative computations.
(3) A script or diary file for your integration. Circle your answer for the shortest path!
(4) A script or diary file for your best fitting curve (Use linear algebra, not the "Basic Fitting" tools!)
(5) A Matlab plot of the final (best) path.